

Measures of Potential Output from an Estimated DSGE Model of the United States

TECHNICAL APPENDIX

(Not for Publication)

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1 INTRODUCTORY NOTES

- Model features unit root shocks to labor-augmenting technology S_t^y and to the inflation target π_t^* .
- To make the model stationary, this requires the following:
 - Scaling of real variables, say x_t : $\check{x}_t = x_t/S_t^y$.
 - Scaling of nominal variables, say Z_t : $z_t = Z_t/P_t^*$ (P_t^* = target price path).
- Rescaling is done throughout this Appendix. We therefore present the unit root processes first.
- Technology Shocks:

- In Levels:

$$S_t^y = S_{t-1}^y g_t$$

$$g_t = g_t^{gr} g_t^{iid}$$

$$\ln g_t^{gr} = (1 - \rho_g) \ln \bar{g} + \rho_g \ln g_{t-1}^{gr} + \hat{\varepsilon}_t^{gr}$$

$$\ln g_t^{iid} = \hat{\varepsilon}_t^{iid}$$

- Linearized:

$$\hat{g}_t = \hat{g}_t^{gr} + \hat{g}_t^{iid} \tag{1}$$

$$\hat{g}_t^{gr} = \rho_g \hat{g}_{t-1}^{gr} + \hat{\varepsilon}_t^{gr} \tag{2}$$

$$\hat{g}_t^{iid} = \hat{\varepsilon}_t^{iid} \tag{3}$$

- Inflation Target Shocks:

- In Levels:

$$\pi_t^* = \pi_{t-1}^* \varepsilon_t^{\pi^*}$$

- Linearized:

$$\hat{\pi}_t^* = \hat{\pi}_{t-1}^* + \hat{\varepsilon}_t^{\pi^*} \tag{4}$$

- Linearization of any variable \check{x}_t around its steady state \bar{x} : $\hat{x}_t = (\check{x}_t - \bar{x})/\bar{x}$.
- All inflation and interest rates are gross rates.
- Indices for different heterogeneous agents:
 - i for households.
 - j for manufacturing firms.
 - z for financial intermediaries.

2 HOUSEHOLDS

2.1 Optimization Problem

- Objective Function for Household i :

$$Max \quad E_0 \sum_{t=0}^{\infty} \beta^t \left\{ S_t^c \left(1 - \frac{v}{\bar{g}}\right) \log(H_t(i)) - S_t^L \psi \frac{L_t(i)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{a}{1-\epsilon} \left(\frac{M_t(i)}{P_t}\right)^{1-\epsilon} \right\}, \text{ where}$$

$$H_t(i) = C_t(i) - \nu C_{t-1} \quad (\text{external habit})$$

$$C_t(i) = \left(\int_0^1 c_t(i, j)^{\frac{\sigma_t^p - 1}{\sigma_t^p}} dj \right)^{\frac{\sigma_t^p}{\sigma_t^p - 1}}, \quad P_t = \left(\int_0^1 P_t(j)^{1 - \sigma_t^p} dj \right)^{\frac{1}{1 - \sigma_t^p}}$$

- Budget Constraint (multiplier = $\lambda_t(i)/P_t$):

$$\begin{aligned} B_t(i) + B_t^D(i) &= i_{t-1} (B_{t-1}(i) + B_{t-1}^D(i)) + M_{t-1}(i) - M_t(i) \\ &\quad + W_t(i)L_t(i) + \int_0^1 \Pi_t(i, j) dj + \int_0^1 \Pi_t(i, z) dz - P_t \tau_t(i) \\ &\quad - P_t C_t(i) - W_t \frac{\phi_w}{2} \frac{(L_t(i) - \ell_t)^2}{\ell_t} \end{aligned}$$

- Cost of deviating from “normal” labor supply of other households:
 - Quadratic in the % deviation from “normal” labor supply $\frac{\phi_w}{2} (L_t(i) - \ell_t)^2 / \ell_t^2$.
 - Proportional to the aggregate wage bill $W_t \ell_t$.
- Assume complete contingent claims markets for labor income and identical initial endowments of capital, bonds and money. Then all FOC are the same except for labor supply. Therefore drop subscript i except in the wage setting problem. The latter is dealt with later, after firm and intermediary price setting.

2.2 First-Order Conditions

- FOC for B_t :

$$\lambda_t = \beta i_t E_t \left(\frac{\lambda_{t+1}}{\pi_{t+1}^p} \right)$$

- Rescaled by technology ($\check{\lambda}_t = \lambda_t S_t^Y$) and by the inflation target ($\check{i}_t = i_t / \pi_t^*$, $\check{\pi}_{t+1}^p = \pi_{t+1}^p / \pi_{t+1}^*$):

$$\check{\lambda}_t = \beta \check{i}_t E_t \left(\frac{\check{\lambda}_{t+1}}{\check{\pi}_{t+1}^p g_{t+1} \epsilon_{t+1}^*} \right)$$

- Steady state ($\bar{r} = \bar{i}/\pi^*$):

$$\bar{r} = \bar{g}/\beta$$

- Linearization (remember that $E_t \hat{c}_{t+1}^{\pi^*} = 0$):

$$\hat{\lambda}_t = \hat{i}_t + E_t \left(\hat{\lambda}_{t+1} - \hat{\pi}_{t+1}^p - \hat{g}_{t+1} \right) \quad (5)$$

- FOC for C_t :

$$\frac{S_t^c (1 - \frac{v}{g})}{\check{H}_t} = \lambda_t$$

- Rescaled by technology ($\check{H}_t = \check{H}_t^t / S_t^y$):

$$\frac{S_t^c (1 - \frac{v}{g})}{\check{H}_t} = \check{\lambda}_t$$

- Steady state:

$$\frac{(1 - \frac{v}{g})}{\bar{H}} = \bar{\lambda}$$

- Linearization:

$$\hat{S}_t^c - \hat{H}_t = \hat{\lambda}_t \quad (6)$$

- Habit H_t :

$$H_t = C_t - \nu C_{t-1}$$

- Rescaled by technology ($\check{C}_t = C_t / S_t^y$):

$$\check{H}_t = \check{C}_t - \nu \frac{\check{C}_{t-1}}{g_t}$$

- Steady state:

$$\bar{H} = \bar{C} (1 - \frac{\nu}{g})$$

- Linearization:

$$\hat{H}_t = \frac{1}{1 - \frac{\nu}{g}} \hat{C}_t - \frac{\frac{\nu}{g}}{1 - \frac{\nu}{g}} (\hat{C}_{t-1} - \hat{g}_t) \quad (7)$$

- Wage Setting: See below, after derivation of price setting.

3 CAPITAL GOODS PRODUCERS

- Investment Decision:

- Investment adjustment costs:

$$G_{I,t} = \frac{\phi_I}{2} I_t \left(\frac{(I_t/g_t) - I_{t-1}}{I_{t-1}} \right)^2$$

- Optimization problem:

$$\underset{\{\check{I}_{t+s}\}_{s=0}^{\infty}}{Max} E_t \sum_{s=0}^{\infty} \beta^s \frac{\lambda_{t+s}}{\lambda_t} \left[q_{t+s} \left(\check{K}_{t+s} + I_{t+s} \right) - q_{t+s} \check{K}_{t+s} - S_{t+s}^{inv} I_{t+s} - G_{I,t+s} \right]$$

- FOC:

$$q_t = S_t^{inv} + \phi_I \left(\frac{\check{I}_t}{\check{I}_{t-1}} \right) \left(\frac{\check{I}_t - \check{I}_{t-1}}{\check{I}_{t-1}} \right) - E_t \beta \frac{\check{\lambda}_{t+1}}{\check{\lambda}_t} \phi_I \left(\frac{\check{I}_{t+1}}{\check{I}_t} \right)^2 \left(\frac{\check{I}_{t+1} - \check{I}_t}{\check{I}_t} \right)$$

- Steady state:

$$\bar{q} = 1$$

- Linearization:

$$\hat{q}_t = \phi_I \left(\hat{I}_t - \hat{I}_{t-1} \right) - \beta \phi_I \left(\hat{I}_{t+1} - \hat{I}_t \right) + \hat{S}_t^{inv} \quad (8)$$

- Capital Accumulation:

$$K_t = (1 - \Delta) K_{t-1} + I_t$$

- Rescaled by technology:

$$\check{K}_t = (1 - \Delta) \frac{\check{K}_{t-1}}{g_t} + \check{I}_t$$

- Steady state:

$$\bar{I} = \frac{(\bar{g} + \Delta - 1)}{\bar{g}} \bar{K}$$

- Linearization:

$$\bar{K} \hat{K}_t = (1 - \Delta) \frac{\bar{K}}{\bar{g}} \left(\hat{K}_{t-1} - \hat{g}_t \right) + \bar{I} \hat{I}_{t-1} \quad (9)$$

4 ENTREPRENEURS

4.1 Capacity Utilization Decision

- Optimization Problem:

$$\underset{u_t}{Max} [u_t r_{k,t} - a(u_t)] \omega_t K_{t-1}(j)$$

- Cost function:

$$a(u_t) = \frac{1}{2} \phi_a \sigma_a (u_t)^2 + \phi_a (1 - \sigma_a) u_t + \phi_a \left(\frac{\sigma_a}{2} - 1 \right)$$

- FOC:

$$r_{k,t} = \phi_a \sigma_a u_t + \phi_a (1 - \sigma_a)$$

- Steady state ($\bar{u} = 1$):

$$\bar{r}_k = \phi_a$$

- Linearization:

$$\hat{r}_{k,t} = \sigma_a \hat{u}_t \tag{10}$$

4.2 Real Return to Utilized Capital

- Definition (nominal return $Ret_{k,t} = E_t (ret_{k,t} \pi_{t+1}^p)$):

$$ret_{k,t} = E_t \frac{u_{t+1} r_{k,t+1} - a(u_{t+1}) + (1 - \Delta) q_{t+1}}{q_t}$$

- Steady state:

$$\overline{ret}_k = 1 - \Delta + \bar{r}_k$$

- Linearization:

$$\widehat{ret}_{k,t} = \frac{1 - \Delta}{1 - \Delta + \bar{r}_k} E_t \hat{q}_{t+1} - \hat{q}_t + \frac{\bar{r}_k}{1 - \Delta + \bar{r}_k} E_t \hat{r}_{k,t+1} \tag{11}$$

4.3 Bank's Zero Profit or Participation Constraint

- Original Constraint:

$$i_t B_t^D(j) = (1 - F(\bar{\omega}_{t+1})) i_{B,t+1} B_t^D(j) + (1 - \xi_{t+1}) \int_0^{\bar{\omega}_{t+1}} Q_t K_t(j) Ret_{k,t} \omega f(\omega) d\omega$$

- i_t = non-contingent interest rate paid by bank to depositors.
- $B_t^D(j)$ = nominal loan amount.
- $\bar{\omega}_{t+1}$ = cutoff level for next period productivity, below = bankruptcy, above = pay loan in full. ($\bar{\omega}_{t+1}$ is a function of $t + 1$ shocks.)
- $i_{B,t+1}$ = gross nominal rate of interest to be paid in full if ω_{t+1} is high enough. ($i_{B,t+1}$ is a function of $t + 1$ shocks.)
- $F(\cdot)$ = cdf of the log-normal distribution of ω , with $E(\omega_t) = 1$ and $Var(\omega_t) = \sigma_t^2$.
- $(1 - \xi_{t+1})$ = fraction of project value that the bank can recover in bankruptcy.
- Cutoff $\bar{\omega}_{t+1}$ condition

$$Ret_{k,t} \bar{\omega}_{t+1} Q_t K_t(j) = i_{B,t+1} B_t^D(j)$$

- Balance sheet constraint ($N_t(j)$ = nominal net worth):

$$Q_t K_t(j) = N_t(j) + B_t^D(j)$$

- Rewritten participation constraint using the foregoing:

$$\begin{aligned} & \left[(1 - F(\bar{\omega}_{t+1})) \bar{\omega}_{t+1} + (1 - \xi_{t+1}) \int_0^{\bar{\omega}_{t+1}} \omega f(\omega) d\omega \right] Ret_{k,t} Q_t K_t(j) \\ & = i_t Q_t K_t(j) - i_t N_t(j) \end{aligned}$$

- Definitions:

- Nominal capital earnings: $Ret_{k,t} Q_t K_t(j)$
- Lender's gross share in capital earnings:

$$\Gamma(\bar{\omega}_{t+1}) \equiv \int_0^{\bar{\omega}_{t+1}} \omega_{t+1} f(\omega_{t+1}) d\omega_{t+1} + \bar{\omega}_{t+1} \int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1}) d\omega_{t+1}$$

- Lender's monitoring costs share in capital earnings:

$$\xi_{t+1} G(\bar{\omega}_{t+1}) = \xi_{t+1} \int_0^{\bar{\omega}_{t+1}} \omega_{t+1} f(\omega_{t+1}) d\omega_{t+1}$$

- Entrepreneur's share in capital earnings

$$1 - \Gamma(\bar{\omega}_{t+1}) = \int_{\bar{\omega}_{t+1}}^{\infty} (\omega_{t+1} - \bar{\omega}_{t+1}) f(\omega_{t+1}) d\omega_{t+1}$$

- Aggregated Participation Constraint (using the foregoing definitions):

$$\frac{q_{t-1}\check{K}_{t-1}}{\check{n}_{t-1}} \frac{r\check{e}t_{k,t}^{m1}}{\check{r}_t^{m1}} (\Gamma(\bar{\omega}_t) - \xi_t G(\bar{\omega}_t)) - \frac{q_{t-1}\check{K}_{t-1}}{\check{n}_{t-1}} + 1 = 0$$

- Note the time subscripts: This has to hold in each state of the world = ex-post.
- Trick to make our timing conventions work:

$$\begin{aligned} r\check{e}t_{k,t}^{m1} &= r\check{e}t_{k,t-1} \\ \check{r}_t^{m1} &= \check{r}_{t-1} \end{aligned}$$

- Steady state:

$$\bar{n} = \bar{K} \left(1 - \frac{\overline{ret}_k}{\bar{r}} (\bar{\Gamma} - \bar{\xi}\bar{G}) \right)$$

- Linearization (superscripts ω and σ indicate partial derivatives, in steady state, with respect to these two variables):

$$\begin{aligned} 0 &= \left(\frac{\bar{K}}{\bar{n}} - 1 \right) \left(\widehat{ret}_{k,t}^{m1} - \widehat{r}_t^{m1} \right) - \left(\widehat{q}_{t-1} + \widehat{K}_{t-1} - \widehat{n}_{t-1} \right) \\ &+ \left(\frac{\bar{K}}{\bar{n}} - 1 \right) \frac{\bar{\Gamma}^\omega - \bar{\xi}\bar{G}^\omega}{\bar{\Gamma} - \bar{\xi}\bar{G}} \bar{\omega}\widehat{\omega}_t + \left(\frac{\bar{K}}{\bar{n}} - 1 \right) \frac{\bar{\Gamma}^\sigma - \bar{\xi}\bar{G}^\sigma}{\bar{\Gamma} - \bar{\xi}\bar{G}} \bar{\sigma}\widehat{\sigma}_t \end{aligned} \quad (12)$$

4.4 Entrepreneur's Optimal Loan Contract

- Entrepreneur's Optimization Problem ($\tilde{\lambda}_t =$ multiplier of the participation constraint):

$$Max_{K_t(j), \bar{\omega}_{t+1}} E_t \{ (1 - \Gamma(\bar{\omega}_{t+1})) Ret_{k,t} Q_t K_t(j) \}$$

$$+ \tilde{\lambda}_t \left[(\Gamma(\bar{\omega}_{t+1}) - \xi_{t+1} G(\bar{\omega}_{t+1})) Ret_{k,t} Q_t K_t(j) - i_t Q_t K_t(j) + i_t N_t(j) \right]$$

- Note the expectations operator: Entrepreneur is risk-neutral and absorbs all aggregate risk.
- Divide through by $i_t N_t(j)$, normalize, and replace nominal by real returns:

$$\begin{aligned} Max_{K_t(j), \bar{\omega}_{t+1}} &\left\{ (1 - \Gamma(\bar{\omega}_{t+1})) \frac{r\check{e}t_{k,t} q_t \check{K}_t(j)}{\check{r}_t \check{n}_t(j)} \right. \\ &\left. + \tilde{\lambda}_t \left[(\Gamma(\bar{\omega}_{t+1}) - \xi_{t+1} G(\bar{\omega}_{t+1})) \frac{r\check{e}t_{k,t} q_t \check{K}_t(j)}{\check{r}_t \check{n}_t(j)} - \frac{q_t \check{K}_t(j)}{\check{n}_t(j)} + 1 \right] \right\} \end{aligned}$$

- FOC with respect to $\bar{\omega}_{t+1}$:

$$-\Gamma_{t+1}^\omega \frac{r\check{e}t_{k,t} q_t \check{K}_t(j)}{\check{r}_t \check{n}_t(j)} + \tilde{\lambda}_t \left\{ (\Gamma_{t+1}^\omega - \xi_{t+1} G_{t+1}^\omega) \frac{r\check{e}t_{k,t} q_t \check{K}_t(j)}{\check{r}_t \check{n}_t(j)} \right\} = 0$$

This implies:

$$\tilde{\lambda}_t = \frac{\Gamma_{t+1}^\omega}{\Gamma_{t+1}^\omega - \xi_{t+1} G_{t+1}^\omega}$$

- Optimal loan contract (FOC with respect to $\check{K}_t(j)$):

$$E_t \left\{ (1 - \Gamma_{t+1}) \frac{r\check{e}t_{k,t}}{\check{r}_t} + \frac{\Gamma_{t+1}^\omega}{\Gamma_{t+1}^\omega - \xi_{t+1} G_{t+1}^\omega} \left[\frac{r\check{e}t_{k,t}}{\check{r}_t} (\Gamma_{t+1} - \xi_{t+1} G_{t+1}) - 1 \right] \right\} = 0$$

– Steady state:

$$\overline{ret}_k = \frac{\bar{r} \bar{\lambda}}{1 - \bar{\Gamma} + \bar{\lambda} (\bar{\Gamma} - \bar{\xi} \bar{G})}$$

– Linearization:

$$\begin{aligned} 0 = & \bar{\lambda} \left(\widehat{ret}_{k,t} - \hat{r}_t \right) - (1 - \bar{\Gamma}) \frac{\bar{r}^k}{\bar{r}} \left(\frac{\bar{\Gamma}^{\omega\omega} - \bar{\lambda} (\bar{\Gamma}^{\omega\omega} - \bar{\xi} \bar{G}^{\omega\omega})}{\bar{\Gamma}^\omega} \right) \bar{\omega} E_t \hat{\omega}_{t+1} \\ & + \frac{\bar{r}^k}{\bar{r}} \left[-\Gamma^\sigma + \bar{\lambda} (\bar{\Gamma}^\sigma - \bar{\xi} \bar{G}^\sigma) - (1 - \bar{\Gamma}) \left(\frac{\bar{\Gamma}^{\omega\sigma} - \bar{\lambda} (\bar{\Gamma}^{\omega\sigma} - \bar{\xi} \bar{G}^{\omega\sigma})}{\bar{\Gamma}^\omega} \right) \right] \bar{\sigma} E_t \hat{\sigma}_{t+1} \end{aligned} \quad (13)$$

4.5 Net Wealth Accumulation

- Nominal net wealth accumulation ($div_t =$ lump-sum dividends to households):

$$N_t = Ret_{k,t-1} Q_{t-1} \check{K}_{t-1} (1 - \xi_t G_t) - i_{t-1} B_{t-1}^D - P_t div_t$$

- Real net wealth accumulation (using the balance sheet identity):

$$\check{n}_t = \check{r}_t^{m1} \frac{\check{n}_{t-1}}{g_t} + \frac{q_{t-1} \check{K}_{t-1}}{g_t} (r\check{e}t_{k,t}^{m1} (1 - \xi_t G_t) - \check{r}_t^{m1}) - \check{d}iv_t$$

- Dividends:

$$\check{d}iv_t = d * \check{n}_t$$

- Steady state:

$$d = \frac{\bar{r}}{\bar{g}} - 1 + \frac{\bar{K}}{\bar{g}\bar{n}} \left[\overline{ret}_k (1 - \bar{\xi} \bar{G}) - \bar{r} \right]$$

- Linearization:

$$\begin{aligned} & \bar{n} \bar{g} (1 + d) (\hat{n}_t + \hat{g}_t) \\ = & \bar{r} (\bar{n} - \bar{K}) \hat{r}_t^{m1} + \bar{r} \bar{n} \hat{n}_{t-1} + \bar{K} \left(\overline{ret}_k (1 - \bar{\xi} \bar{G}) - \bar{r} \right) (\hat{q}_{t-1} + \hat{K}_{t-1}) \\ & + \bar{K} \overline{ret}_k (1 - \bar{\xi} \bar{G}) \left(\widehat{ret}_{k,t}^{m1} \right) - \bar{K} \overline{ret}_k \bar{\xi} (\bar{G}^\omega \bar{\omega} \hat{\omega}_t + \bar{G}^\sigma \bar{\sigma} \hat{\sigma}_t) - \bar{K} \overline{ret}_k \bar{G} \hat{\xi}_t \end{aligned}$$

4.6 Closed Forms for Γ , G and Their Partial Derivatives

4.6.1 Basic Properties of Γ and G

- Repeat expressions for Γ and G for ease of reference:

$$\Gamma(\bar{\omega}_{t+1}) \equiv \int_0^{\bar{\omega}_{t+1}} \omega_{t+1} f(\omega_{t+1}) d\omega_{t+1} + \bar{\omega}_{t+1} \int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1}) d\omega_{t+1}$$

$$G(\bar{\omega}_{t+1}) = \int_0^{\bar{\omega}_{t+1}} \omega_{t+1} f(\omega_{t+1}) d\omega_{t+1}$$

- First derivatives with respect to $\bar{\omega}_{t+1}$:

$$\Gamma_{t+1}^{\omega} = 1 - F(\bar{\omega}_{t+1})$$

$$G_{t+1}^{\omega} = \bar{\omega}_{t+1} f(\bar{\omega}_{t+1})$$

4.6.2 Basic Properties of the Lognormal Distribution

The assumption is that ω_t is lognormally distributed with $E(\omega_t) = 1$ and $Var(\omega_t) = \sigma_t^2$. This implies the following:

$$\ln(\omega_t) \sim N\left(-\frac{1}{2}\sigma_t^2, \sigma_t^2\right)$$

$$f(\omega_t) = \frac{1}{\sqrt{2\pi}\omega_t\sigma_t} \exp\left\{-\frac{1}{2}\left(\frac{\ln(\omega_t) + \frac{1}{2}\sigma_t^2}{\sigma_t}\right)^2\right\}$$

4.6.3 Derivations A: The Nonlinear System

- We will change integrands at various points in order to obtain solutions that can be expressed in terms of the cumulative distribution function Φ of the standard normal distribution.
- First define terms:

$$\bar{z}_t = \frac{\ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t}, \quad y_t = \frac{\ln(\omega_t) + \frac{1}{2}\sigma_t^2}{\sigma_t}$$

$$\tilde{z}_t = \frac{\ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t}, \quad \tilde{y}_t = \frac{\ln(\omega_t) - \frac{1}{2}\sigma_t^2}{\sigma_t}$$

- Manipulating the second expression in each case gives:

$$d\omega_t = \sigma_t \exp\left\{y_t \sigma_t - \frac{1}{2}\sigma_t^2\right\} dy_t$$

$$d\omega_t = \sigma_t \exp\left\{\tilde{y}_t \sigma_t + \frac{1}{2}\sigma_t^2\right\} d\tilde{y}_t$$

- Now evaluate the expressions determining Γ and G in terms of the c.d.f. $\Phi(\cdot)$:

– First Component:

$$\begin{aligned}
\int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1}) d\omega_{t+1} &= \int_{\bar{\omega}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}\omega_{t+1}\sigma_{t+1}} \exp\left\{-\frac{1}{2}\left(\frac{\ln(\omega_{t+1}) + \frac{1}{2}\sigma_{t+1}^2}{\sigma_{t+1}}\right)^2\right\} d\omega_{t+1} \\
&= \int_{\bar{z}_{t+1}}^{\infty} \frac{\sigma_{t+1}}{\sqrt{2\pi}\omega_{t+1}\sigma_{t+1}} \exp\left\{-\frac{1}{2}y_{t+1}^2\right\} \exp\left\{y_{t+1}\sigma_{t+1} - \frac{1}{2}\sigma_{t+1}^2\right\} dy_{t+1} \\
&= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\omega_{t+1}} \exp\left\{-\frac{1}{2}(y_{t+1}^2 + \sigma_{t+1}^2 - 2y_{t+1}\sigma_{t+1})\right\} dy_{t+1} \\
&= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\omega_{t+1}} \exp\left\{-\frac{1}{2}(y_{t+1} - \sigma_{t+1})^2\right\} dy_{t+1} \\
&= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\ln(\omega_{t+1})\} \exp\left\{-\frac{(\ln(\omega_{t+1}) - \frac{1}{2}\sigma_{t+1}^2)^2}{2\sigma_{t+1}^2}\right\} dy_{t+1} \\
&= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-2\sigma_{t+1}^2 \ln(\omega_{t+1}) - (\ln(\omega_{t+1}))^2 - (\frac{1}{2}\sigma_{t+1}^2)^2 + \ln(\omega_{t+1})\sigma_{t+1}^2}{2\sigma_{t+1}^2}\right\} dy_{t+1} \\
&= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(\ln(\omega_{t+1}))^2 + (\frac{1}{2}\sigma_{t+1}^2)^2 + 2\ln(\omega_{t+1})\frac{1}{2}\sigma_{t+1}^2}{2\sigma_{t+1}^2}\right\} dy_{t+1} \\
&= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln(\omega_{t+1}) + \frac{1}{2}\sigma_{t+1}^2}{\sigma_{t+1}}\right)^2\right\} dy_{t+1} \\
&= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(y_{t+1})^2\right\} dy_{t+1} = 1 - \Phi(\bar{z}_{t+1})
\end{aligned}$$

– Second Component:

$$\begin{aligned}
\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1} f(\omega_{t+1}) d\omega_{t+1} &= \int_{\bar{\omega}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{t+1}} \exp\left\{-\frac{1}{2}\left(\frac{\ln(\omega_{t+1}) + \frac{1}{2}\sigma_{t+1}^2}{\sigma_{t+1}}\right)^2\right\} d\omega_{t+1} \\
&= \int_{\bar{z}_{t+1}}^{\infty} \frac{\sigma_{t+1}}{\sqrt{2\pi}\sigma_{t+1}} \exp\left\{-\frac{1}{2}(\tilde{y}_{t+1} + \sigma_{t+1})^2\right\} \exp\left\{\tilde{y}_{t+1}\sigma_{t+1} + \frac{1}{2}\sigma_{t+1}^2\right\} d\tilde{y}_{t+1} \\
&= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\tilde{y}_{t+1}^2 - \frac{1}{2}\sigma_{t+1}^2 - \tilde{y}_{t+1}\sigma_{t+1} + \tilde{y}_{t+1}\sigma_{t+1} + \frac{1}{2}\sigma_{t+1}^2\right\} d\tilde{y}_{t+1} \\
&= \int_{\bar{z}_{t+1}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\tilde{y}_{t+1}^2\right\} d\tilde{y}_{t+1} = 1 - \Phi(\bar{z}_{t+1}) = 1 - \Phi(\bar{z}_{t+1} - \sigma_{t+1})
\end{aligned}$$

– Summary:

$$\int_{\bar{\omega}_{t+1}}^{\infty} f(\omega_{t+1})d\omega_{t+1} = 1 - \Phi(\bar{z}_{t+1})$$

$$\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1}f(\omega_{t+1})d\omega_{t+1} = 1 - \Phi(\bar{z}_{t+1} - \sigma_{t+1})$$

– Final set of nonlinear equations:

$$\bar{z}_t = \frac{\ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t}$$

$$f(\bar{\omega}_t) = \frac{1}{\sqrt{2\pi\bar{\omega}_t\sigma_t}} \exp\left\{-\frac{1}{2}\bar{z}_t^2\right\}$$

$$\Gamma_t = \Phi(\bar{z}_t - \sigma_t) + \bar{\omega}_t(1 - \Phi(\bar{z}_t))$$

$$G_t = \Phi(\bar{z}_t - \sigma_t)$$

$$\Gamma_t^\omega = 1 - \Phi(\bar{z}_t)$$

$$G_t^\omega = \bar{\omega}_t f(\bar{\omega}_t)$$

4.6.4 Derivations B: Derivatives for the Linearized System

- Closed Form Expressions for Standard Normal cdf's:

$$\Phi(\bar{z}_t - \sigma_t) = \int_0^{\bar{z}_t - \sigma_t} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}y_t^2\right\} dy_t$$

$$\Phi(\bar{z}_t) = \int_0^{\bar{z}_t} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}y_t^2\right\} dy_t$$

- Auxiliary Relationships:

– Derivatives of Γ :

* We have the following:

$$\Gamma_t = \Phi(\bar{z}_t - \sigma_t) + \bar{\omega}_t(1 - \Phi(\bar{z}_t))$$

$$\Gamma_t^\omega = 1 - \Phi(\bar{z}_t)$$

* We take the derivative of the first expression:

$$\Gamma_t^\omega = \Phi'(\bar{z}_t - \sigma_t) \frac{1}{\bar{\omega}_t\sigma_t} + (1 - \Phi(\bar{z}_t)) - \bar{\omega}_t\Phi'(\bar{z}_t) \frac{1}{\bar{\omega}_t\sigma_t}$$

* We equate this to the first expression to obtain:

$$\Phi'(\bar{z}_t - \sigma_t) = \bar{\omega}_t\Phi'(\bar{z}_t)$$

– Derivatives of G :

* We have the following:

$$G_t = \Phi(\bar{z}_t - \sigma_t)$$

$$G_t^\omega = \bar{\omega}_t f(\bar{\omega}_t)$$

* We take the derivative of the first expression:

$$G_t^\omega = \Phi'(\bar{z}_t - \sigma_t) \frac{1}{\bar{\omega}_t \sigma_t}$$

* We combine this with the second expression to obtain our final two closed form relationships for derivatives of standard normal cdf's:

$$\Phi'(\bar{z}_t - \sigma_t) = \bar{\omega}_t^2 \sigma_t f(\bar{\omega}_t)$$

$$\Phi'(\bar{z}_t) = \bar{\omega}_t \sigma_t f(\bar{\omega}_t)$$

• $\Gamma_t^{\omega\omega}$:

$$\Gamma_t^{\omega\omega} = -\Phi'(\bar{z}_t) \frac{1}{\bar{\omega}_t \sigma_t} = -\frac{1}{\sqrt{2\pi\bar{\omega}_t \sigma_t}} \exp\left\{-\frac{1}{2}\bar{z}_t^2\right\} = -f(\bar{\omega}_t)$$

• $\Gamma_t^{\omega\sigma}$:

$$\begin{aligned} \Gamma_t^{\omega\sigma} &= -\Phi'(\bar{z}_t) \frac{\sigma_t^2 - \ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t^2} \\ &= -\frac{1}{\sqrt{2\pi\bar{\omega}_t \sigma_t}} \exp\left\{-\frac{1}{2}\bar{z}_t^2\right\} \bar{\omega}_t \frac{\ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t} \\ &= f(\bar{\omega}_t) \bar{\omega}_t (\bar{z}_t - \sigma_t) \end{aligned}$$

• Γ_t^σ :

$$\begin{aligned} \Gamma_t^\sigma &= \Phi'(\bar{z}_t - \sigma_t) \frac{-\sigma_t^2 - \ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t^2} - \bar{\omega}_t \Phi'(\bar{z}_t) \frac{\sigma_t^2 - \ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t^2} \\ &= \Phi'(\bar{z}_t - \sigma_t) \left(\frac{-\ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t^2} - \frac{-\ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t^2} \right) = -\Phi'(\bar{z}_t - \sigma_t) \\ &= -\bar{\omega}_t^2 \sigma_t f(\bar{\omega}_t) \end{aligned}$$

• $G_t^{\omega\omega}$:

$$\begin{aligned} G_t^{\omega\omega} &= f(\bar{\omega}_t) + \bar{\omega}_t \left[-\frac{1}{\sqrt{2\pi\bar{\omega}_t \sigma_t}} \frac{1}{\bar{\omega}_t} \exp\left\{-\frac{1}{2}\bar{z}_t^2\right\} + \frac{1}{\sqrt{2\pi\bar{\omega}_t \sigma_t}} (-\bar{z}_t) \exp\left\{-\frac{1}{2}\bar{z}_t^2\right\} \frac{1}{\bar{\omega}_t \sigma_t} \right] \\ &= f(\bar{\omega}_t) + \bar{\omega}_t f(\bar{\omega}_t) \left[-\frac{1}{\bar{\omega}_t} - \frac{\bar{z}_t}{\bar{\omega}_t \sigma_t} \right] = f(\bar{\omega}_t) \left[1 - 1 - \frac{\bar{z}_t}{\sigma_t} \right] \\ &= -f(\bar{\omega}_t) \frac{\bar{z}_t}{\sigma_t} \end{aligned}$$

- $G_t^{\omega\sigma}$:

$$\begin{aligned}
G_t^{\omega\sigma} &= \bar{\omega}_t \left[-\frac{1}{\sqrt{2\pi}\bar{\omega}_t\sigma_t} \frac{1}{\sigma_t} \exp\left\{-\frac{1}{2}\bar{z}_t^2\right\} + \frac{1}{\sqrt{2\pi}\bar{\omega}_t\sigma_t} (-\bar{z}_t) \exp\left\{-\frac{1}{2}\bar{z}_t^2\right\} \frac{\sigma_t^2 - \ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t^2} \right] \\
&= \bar{\omega}_t f(\bar{\omega}_t) \left[-\frac{1}{\sigma_t} + \frac{1}{\sigma_t} \bar{z}_t \frac{\ln(\bar{\omega}_t) - \frac{1}{2}\sigma_t^2}{\sigma_t} \right] \\
&= \frac{\bar{\omega}_t}{\sigma_t} f(\bar{\omega}_t) [\bar{z}_t (\bar{z}_t - \sigma_t) - 1]
\end{aligned}$$

- G_t^σ :

$$\begin{aligned}
G_t^\sigma &= \Phi'(\bar{z}_t - \sigma_t) \frac{-\sigma_t^2 - \ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t^2} = -\bar{\omega}_t^2 \sigma_t f(\bar{\omega}_t) \frac{\ln(\bar{\omega}_t) + \frac{1}{2}\sigma_t^2}{\sigma_t^2} \\
&= -\bar{\omega}_t^2 f(\bar{\omega}_t) \bar{z}_t
\end{aligned}$$

4.6.5 Standard Normal cdf's and Complementary Error Functions

- General Relationship:

$$\Phi(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{-x}{\sqrt{2}}\right)$$

- Applied to our case:

$$\Phi(\bar{z}_t - \sigma_t) = \frac{1}{2} \operatorname{erfc}\left(\frac{\sigma_t - \bar{z}_t}{\sqrt{2}}\right)$$

$$\Phi(\bar{z}_t) = \frac{1}{2} \operatorname{erfc}\left(\frac{-\bar{z}_t}{\sqrt{2}}\right)$$

5 FIRMS

5.1 Cost Minimization

- Production Functions:

$$y_t(j) = (S_t^y \ell_t(j))^{1-\alpha} k_t(j)^\alpha$$

- Real Marginal Cost:

- In levels:

$$mc_t = \frac{A w_t^{1-\alpha} (r_t^k)^\alpha}{(S_t^y)^{1-\alpha}}, \quad \text{where } A = \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)}$$

- Rescaled by technology ($\check{w}_t = w_t/S_t^y$):

$$mc_t = A \check{w}_t^{1-\alpha} (r_t^k)^\alpha$$

- Linearized:

$$\widehat{mc}_t = (1-\alpha)\hat{w}_t + \alpha\hat{r}_t^k \quad (14)$$

- Input Demands for Aggregate Firm Sector (see next page for definitions of aggregate variables):

- In levels:

$$\ell_t = (1-\alpha) \frac{mc_t}{w_t} \tilde{Y}_t$$

$$k_t = \alpha \frac{mc_t}{r_t^k} \tilde{Y}_t$$

- Rescaled by technology ($\check{Y}_t = \tilde{Y}_t/S_t^y$):

$$\ell_t = (1-\alpha) \frac{mc_t}{\check{w}_t} \check{Y}_t$$

$$\check{k}_t = \alpha \frac{mc_t}{r_t^k} \check{Y}_t$$

- Linearized:

$$\hat{\ell}_t = \widehat{mc}_t - \hat{w}_t + \hat{Y}_t \quad (15)$$

$$\hat{k}_t = \widehat{mc}_t - \hat{r}_t^k + \hat{Y}_t \quad (16)$$

- Relationship between physical capital K_t and utilized capital k_t :

- In levels:

$$k_t = u_t K_{t-1}$$

- Rescaled by technology:

$$\check{k}_t = u_t \frac{\check{K}_{t-1}}{g_t}$$

- Steady state:

$$\bar{k} = \frac{\bar{K}}{\bar{g}}$$

- Linearization:

$$\hat{k}_t = \hat{K}_{t-1} + \hat{u}_t - \hat{g}_t$$

- Definitions used above:

- Labor:

- * Aggregate:

$$\ell_t = \int_0^1 \ell_t(j) dj \quad , \quad \text{where } \ell_t(j) = \left(\int_0^1 L_t(j, i)^{\frac{\sigma_t^w - 1}{\sigma_t^w}} di \right)^{\frac{\sigma_t^w}{\sigma_t^w - 1}}$$

$$L_t(i) = \int_0^1 L_t(j, i) dj$$

- * Varieties $L_t(j, i)$ supplied by households i , see above and also Section 5 on household wage setting.

- * Cost minimizing varieties demands:

$$L_t(j, i) = \ell_t(j) \left(\frac{W_t(i)}{W_t} \right)^{-\sigma_t^w} \quad , \quad \text{therefore } L_t(i) = \ell_t \left(\frac{W_t(i)}{W_t} \right)^{-\sigma_t^w}$$

- Capital:

$$k_t = \int_0^1 k_t(j) dj$$

- Output:

- * Aggregate:

$$\tilde{Y}_t = \int_0^1 y_t(j) dj \quad , \quad \text{while } Y_t = \left(\int_0^1 y_t(j)^{\frac{\sigma_t^p - 1}{\sigma_t^p}} dj \right)^{\frac{\sigma_t^p}{\sigma_t^p - 1}}$$

- * It is easy to show that:

$$\bar{\tilde{Y}} = \bar{Y} \quad , \quad \hat{\tilde{Y}}_t = \hat{Y}_t$$

5.2 Profit Maximization

- Discounted real profits:
 - Real revenue $\frac{P_{t+k}(j)}{P_{t+k}} y_{t+k}(j)$.
 - Real marginal cost $\frac{MC_{t+k}}{P_{t+k}} y_{t+k}(j)$.
 - Real cost of deviating from “normal” output of other firms:
 - * Quadratic in the % deviation from “normal” output $\frac{\phi_p}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}^2}$.
 - * Proportional to aggregate real output Y_{t+k} .
- Pricing policy of firm j that reoptimizes at t , choosing V_t^p and v_t^p (a gross inflation rate):

$$P_{t+k}(j) = V_t^p (v_t^p)^k$$

- Profit maximization:

$$\begin{aligned} \underset{V_t^p, v_t^p}{Max} E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \lambda_{t+k} & \left[\frac{V_t^p (v_t^p)^k}{P_{t+k}} y_{t+k}(j) - mc_{t+k} y_{t+k}(j) - \frac{\phi_p}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}^2} \right], \text{ s.t.} \\ y_{t+k}(j) & = Y_{t+k} \left(\frac{V_t^p (v_t^p)^k}{P_{t+k}} \right)^{-\sigma_{t+k}^p} \end{aligned}$$

- Substitute constraints:

$$\underset{V_t^p, v_t^p}{Max} E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \lambda_{t+k} \left[\left(\frac{V_t^p (v_t^p)^k}{P_{t+k}} \right)^{1-\sigma_{t+k}^p} Y_{t+k} - mc_{t+k} \left(\frac{V_t^p (v_t^p)^k}{P_{t+k}} \right)^{-\sigma_{t+k}^p} Y_{t+k} - \frac{\phi_p}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}^2} \right]$$

- Define terms:

- Front-loading term:

$$p_t^p \equiv \frac{V_t^p}{P_t}$$

- Inflation rescaled by the inflation target:

$$\tilde{\pi}_t^p = \pi_t^p / \pi_t^*$$

- Cumulative aggregate rescaled inflation:

$$\tilde{\Pi}_{t,k}^p \equiv \prod_{j=1}^k \tilde{\pi}_{t+j}^p \text{ for } k \geq 1 \quad (\equiv 1 \text{ for } k = 0) \quad (17)$$

- Cumulative aggregate rescaled inflation deviation:

$$\hat{\Pi}_{t,k}^p \equiv \sum_{j=1}^k \hat{\pi}_{t+j}^p \text{ for } k \geq 1 \quad (\equiv 0 \text{ for } k = 0) \quad (18)$$

- Mark-up:

$$\mu_t^p = \frac{\sigma_t^p}{\sigma_t^p - 1}$$

5.3 First-Order Conditions

- Rescaled by technology, with $\check{y}_t(j) = y_t(j)/S_t^y$.
- Rescaled by the inflation target, with $\check{v}_t^p = v_t^p/\pi_t^*$.
- FOC for V_t^p :

$$p_t^p = \frac{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k}^p \left(mc_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right)}{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) (\sigma_{t+k}^p - 1) \left(\frac{(\check{v}_t^p)^k}{\check{\Pi}_{t,k}^p} \right)} \quad (19)$$

- FOC w.r.t. v_t^p (rescaled by technology):

$$p_t^p = \frac{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k}^p \left(mc_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right)}{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k \check{\lambda}_{t+k} \check{y}_{t+k}(j) (\sigma_{t+k}^p - 1) \left(\frac{(\check{v}_t^p)^k}{\check{\Pi}_{t,k}^p} \right)} \quad (20)$$

- Rescaling by the inflation target - comments:

- This takes the form of dividing all price levels by the target price level P_t^* .
- In the preceding formulas this only affects $E_t \frac{(v_t^p)^k}{\Pi_{t,k}^p}$, i.e. future firm-specific and aggregate cumulative inflation rates have to be deflated by future cumulative target inflation rates.
- However, under unit roots all **expected** future quarterly target inflation rates are simply equal to the current target rate.
- In all **forward-looking** conditions like (19) and (20) we can therefore simply linearize around the current inflation target π_t^* .
- Example: Our final conditions will contain **expected** inflation terms for periods t and $t + 1$. For actual future inflation the correct definition is $\hat{\pi}_{t+1}^p = \ln(\pi_{t+1}^p) - \ln(\pi_{t+1}^*)$. But it is ok to linearize around π_t^* instead because only the **expected** future inflation deviation enters the equations, and for this we have (up to an approximation error that disappears in linearization):

$$E_t \hat{\pi}_{t+1}^p = E_t (\ln(\pi_{t+1}^p) - \ln(\pi_{t+1}^*)) = E_t (\ln(\pi_{t+1}^p) - \ln(\pi_t^*))$$

- The same is not true for **backward-looking** conditions. See Section 6 on “The Price Index”.

5.4 Linearization

- Geometric distribution formulas:

$$\sum_{k=0}^{\infty} (\delta_p \beta)^k k = \frac{\delta_p \beta}{(1 - \delta_p \beta)^2} \quad (21)$$

$$\sum_{k=0}^{\infty} (\delta_p \beta)^k k^2 = \frac{\delta_p \beta (1 + \delta_p \beta)}{(1 - \delta_p \beta)^3} \quad (22)$$

- Linearization of goods demand:

$$\left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k} \right) = -\bar{\sigma}_p \left(\hat{p}_t^p + k \hat{v}_t^p - \hat{\Pi}_{t,k}^p \right) \quad (23)$$

- Linearization of markup:

$$\hat{\mu}_t^p = \hat{\sigma}_t^p - \bar{\mu}_p \hat{\sigma}_t^p \quad (24)$$

5.4.1 Linearization for V_t^p

- Rewrite (19):

$$\begin{aligned} & \hat{p}_t^p (\sigma_{t+k}^p - 1) E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \left(\frac{(\check{v}_t^p)^k}{\check{\Pi}_{t,k}^p} \right) \\ &= E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k}^p \left(m c_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right) \end{aligned}$$

- Linearization of the common terms $\check{\lambda}_{t+k} \check{y}_{t+k}(j)$ cancels.
- Remaining terms are:

$$\begin{aligned} & \frac{\hat{p}_t^p}{1 - \delta_p \beta} + E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \left[k \hat{v}_t^p - \hat{\Pi}_{t,k}^p + \bar{\mu}_p \hat{\sigma}_{t+k}^p \right] \\ &= E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \left[\widehat{m c}_{t+k} + \phi_p \bar{\mu}_p \left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k} \right) + \hat{\sigma}_{t+k}^p \right] \end{aligned}$$

- Using (21), (23) and (24):

$$\frac{\hat{p}_t^p}{1 - \delta_p \beta} (1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) + \frac{\hat{v}_t^p \delta_p \beta}{(1 - \delta_p \beta)^2} (1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) = E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \left[\widehat{m c}_{t+k} + \hat{\Pi}_{t,k}^p (1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) + \hat{\mu}_{t+k}^p \right]$$

- Equivalently:

$$\frac{\hat{p}_t^p}{1 - \delta_p \beta} + \frac{\hat{v}_t^p \delta_p \beta}{(1 - \delta_p \beta)^2} = E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \left[\frac{\widehat{m c}_{t+k} + \hat{\mu}_{t+k}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\Pi}_{t,k}^p \right] \quad (25)$$

5.4.2 Linearization for v_t^p

- Rewrite (20):

$$\begin{aligned} & p_t^p E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k \check{\lambda}_{t+k} \check{y}_{t+k}(j) (\sigma_{t+k}^p - 1) \left(\frac{(\check{v}_t^p)^k}{\check{\Pi}_{t,k}} \right) \\ &= E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k}^p \left(m c_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right) \end{aligned}$$

- Linearization of the common terms $\check{\lambda}_{t+k} \check{y}_{t+k}(j)$ cancels.
- Applying (21) to the \hat{p}_t^p -term, the remaining terms are:

$$\begin{aligned} & \frac{\hat{p}_t^p \delta_p \beta}{(1 - \delta_p \beta)^2} + E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k \left[k \hat{v}_t^p - \hat{\Pi}_{t,k}^p + \bar{\mu}_p \hat{\sigma}_{t+k}^p \right] \\ &= E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k \left[\widehat{m} c_{t+k} + \phi_p \bar{\mu}_p \left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k} \right) + \hat{\sigma}_{t+k}^p \right] \end{aligned}$$

- Applying (23) and (24), and simplifying, we get:

$$\begin{aligned} & \frac{\hat{p}_t^p \delta_p \beta}{(1 - \delta_p \beta)^2} (1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) + E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k^2 \hat{v}_t^p (1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) \\ &= E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k \left[\widehat{m} c_{t+k} + \hat{\mu}_{t+k}^p + \hat{\Pi}_{t,k}^p (1 + \phi_p \bar{\mu}_p \bar{\sigma}_p) \right] \end{aligned}$$

- Applying (22) to the \hat{v}_t^p -term and simplifying further, we get:

$$\frac{\hat{p}_t^p \delta_p \beta}{(1 - \delta_p \beta)^2} + \frac{\hat{v}_t^p \delta_p \beta (1 + \delta_p \beta)}{(1 - \delta_p \beta)^3} = E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k \left[\frac{\widehat{m} c_{t+k} + \hat{\mu}_{t+k}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\Pi}_{t,k}^p \right] \quad (26)$$

5.5 Quasi-Differencing

5.5.1 Quasi-Differencing for V_t^p -Equation (25)

- Rewrite (25) as:

$$\hat{p}_t^p + \frac{\hat{v}_t^p \delta_p \beta}{(1 - \delta_p \beta)} = (1 - \delta_p \beta) E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \left[\frac{\widehat{m}c_{t+k} + \hat{\mu}_{t+k}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\Pi}_{t,k}^p \right] \quad (27)$$

- For future reference, lead this by one period and multiply by $(1 - \delta_p \beta)$:

$$\begin{aligned} & (1 - \delta_p \beta) E_t \hat{p}_{t+1}^p + \delta_p \beta E_t \hat{v}_{t+1}^p \\ &= (1 - \delta_p \beta)^2 E_t \left[\sum_{k=0}^{\infty} (\delta_p \beta)^k \frac{\widehat{m}c_{t+1+k} + \hat{\mu}_{t+1+k}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \sum_{k=1}^{\infty} (\delta_p \beta)^k \hat{\Pi}_{t+1,k}^p \right] \end{aligned} \quad (28)$$

- Note the following for $\hat{\Pi}_{t+1,k}^p$:¹

$$\begin{aligned} \hat{\Pi}_{t+1,k}^p &= 0 \text{ for } k = 0, \\ &= \hat{\pi}_{t+2}^p \text{ for } k=1 \\ &= \hat{\pi}_{t+2}^p + \dots + \hat{\pi}_{t+k+1}^p \text{ for } k=2,3,4,\dots \end{aligned} \quad (29)$$

- Write out terms in (27):

$$\begin{aligned} & \hat{p}_t^p + \frac{\hat{v}_t^p \delta_p \beta}{(1 - \delta_p \beta)} = (1 - \delta_p \beta) * \\ E_t & \left[\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + (\delta_p \beta) \left(\frac{\widehat{m}c_{t+1} + \hat{\mu}_{t+1}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+1}^p \right) + (\delta_p \beta)^2 \left(\frac{\widehat{m}c_{t+2} + \hat{\mu}_{t+2}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+1}^p + \hat{\pi}_{t+2}^p \right) \right. \\ & \left. + (\delta_p \beta)^3 \left(\frac{\widehat{m}c_{t+3} + \hat{\mu}_{t+3}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+1}^p + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p \right) + \dots \right] \end{aligned}$$

- Multiply the last equation by $\delta_p \beta$, and lead by one period:

$$\begin{aligned} & \delta_p \beta E_t \hat{p}_{t+1}^p + \frac{(\delta_p \beta)^2}{(1 - \delta_p \beta)} E_t \hat{v}_{t+1}^p = (1 - \delta_p \beta) * \\ E_t & \left[(\delta_p \beta) \frac{\widehat{m}c_{t+1} + \hat{\mu}_{t+1}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + (\delta_p \beta)^2 \left(\frac{\widehat{m}c_{t+2} + \hat{\mu}_{t+2}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+2}^p \right) \right. \\ & \left. + (\delta_p \beta)^3 \left(\frac{\widehat{m}c_{t+3} + \hat{\mu}_{t+3}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p \right) + \dots \right] \end{aligned}$$

¹ The first line shows why for the last term in the previous equation we can let the subscript run from 1 instead of 0.

- Deduct the last equation from the preceding one:

$$\begin{aligned} & [\hat{p}_t^p - \delta_p \beta E_t \hat{p}_{t+1}^p] + \frac{\delta_p \beta}{(1 - \delta_p \beta)} [\hat{v}_t^p - \delta_p \beta E_t \hat{v}_{t+1}^p] \\ &= (1 - \delta_p \beta) \left[\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + E_t \hat{\pi}_{t+1}^p (\delta_p \beta + (\delta_p \beta)^2 + \dots + (\delta_p \beta)^3 + \dots) \right] \end{aligned}$$

- Equivalently:

$$\begin{aligned} & [\hat{p}_t^p - E_t \hat{p}_{t+1}^p + (1 - \delta_p \beta) E_t \hat{p}_{t+1}^p] + \frac{\delta_p \beta}{(1 - \delta_p \beta)} [\hat{v}_t^p - E_t \hat{v}_{t+1}^p + (1 - \delta_p \beta) E_t \hat{v}_{t+1}^p] \\ &= (1 - \delta_p \beta) \frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \delta_p \beta E_t \hat{\pi}_{t+1}^p \end{aligned}$$

- Equivalently:

$$\begin{aligned} & [\hat{p}_t^p - E_t \hat{p}_{t+1}^p] + \frac{\delta_p \beta}{(1 - \delta_p \beta)} [\hat{v}_t^p - E_t \hat{v}_{t+1}^p] \\ &= (1 - \delta_p \beta) \left(\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} - E_t \hat{p}_{t+1}^p \right) + \delta_p \beta (E_t \hat{\pi}_{t+1}^p - E_t \hat{v}_{t+1}^p) \end{aligned}$$

- Equivalently:

$$\begin{aligned} & [E_t \hat{p}_{t+1}^p - \hat{p}_t^p] + \frac{\delta_p \beta}{(1 - \delta_p \beta)} [E_t \hat{v}_{t+1}^p - \hat{v}_t^p] \\ &= -(1 - \delta_p \beta) \left(\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} - E_t \hat{p}_{t+1}^p \right) + \delta_p \beta (E_t \hat{v}_{t+1}^p - E_t \hat{\pi}_{t+1}^p) \end{aligned}$$

- Equivalently:

$$\begin{aligned} & \delta_p \beta E_t \hat{p}_{t+1}^p - \hat{p}_t^p + \frac{\delta_p \beta}{(1 - \delta_p \beta)} [E_t \hat{v}_{t+1}^p - \hat{v}_t^p] \tag{30} \\ &= -(1 - \delta_p \beta) \left(\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \right) + \delta_p \beta E_t \hat{v}_{t+1}^p - \delta_p \beta E_t \hat{\pi}_{t+1}^p \end{aligned}$$

5.5.2 Quasi-Differencing for v_t^p -Equation (26)

- Rewrite (26) as:

$$\hat{p}_t^p + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} \hat{v}_t^p = \frac{(1 - \delta_p \beta)^2}{\delta_p \beta} E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k k \left[\frac{\widehat{m}c_{t+k} + \hat{\mu}_{t+k}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\Pi}_{t,k}^p \right]$$

- Write out terms:

$$\begin{aligned} \hat{p}_t^p + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} \hat{v}_t^p &= \left(\frac{(1 - \delta_p \beta)^2}{\delta_p \beta} \right) * \\ E_t \left[(\delta_p \beta) \left(\frac{\widehat{m}c_{t+1} + \hat{\mu}_{t+1}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+1}^p \right) + 2 (\delta_p \beta)^2 \left(\frac{\widehat{m}c_{t+2} + \hat{\mu}_{t+2}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+1}^p + \hat{\pi}_{t+2}^p \right) \right. \\ &\quad \left. + 3 (\delta_p \beta)^3 \left(\frac{\widehat{m}c_{t+3} + \hat{\mu}_{t+3}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+1}^p + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p \right) + \dots \right] \end{aligned}$$

- Multiply the last equation by $\delta_p \beta$, and lead by one period:

$$\begin{aligned} \delta_p \beta E_t \hat{p}_{t+1}^p + \frac{\delta_p \beta (1 + \delta_p \beta)}{(1 - \delta_p \beta)} E_t \hat{v}_{t+1}^p &= \left(\frac{(1 - \delta_p \beta)^2}{\delta_p \beta} \right) * \\ E_t \left[(\delta_p \beta)^2 \left(\frac{\widehat{m}c_{t+2} + \hat{\mu}_{t+2}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+2}^p \right) + 2 (\delta_p \beta)^3 \left(\frac{\widehat{m}c_{t+3} + \hat{\mu}_{t+3}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p \right) + \dots \right] \end{aligned}$$

- Deduct the last equation from the preceding one:

$$\begin{aligned} [\hat{p}_t^p - \delta_p \beta E_t \hat{p}_{t+1}^p] + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} [\hat{v}_t^p - \delta_p \beta E_t \hat{v}_{t+1}^p] &= \left(\frac{(1 - \delta_p \beta)^2}{\delta_p \beta} \right) * \\ E_t \left[(\delta_p \beta) \frac{\widehat{m}c_{t+1} + \hat{\mu}_{t+1}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + (\delta_p \beta)^2 \left(\frac{\widehat{m}c_{t+2} + \hat{\mu}_{t+2}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+2}^p \right) + (\delta_p \beta)^3 \left(\frac{\widehat{m}c_{t+3} + \hat{\mu}_{t+3}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p \right) + \dots \right. \\ &\quad \left. + \hat{\pi}_{t+1}^p (\delta_p \beta + 2 (\delta_p \beta)^2 + 3 (\delta_p \beta)^3 + \dots) \right] \end{aligned}$$

- Use (21) for the final term:

$$E_t \hat{\pi}_{t+1}^p \left(\frac{(1 - \delta_p \beta)^2}{\delta_p \beta} \right) (\delta_p \beta + 2 (\delta_p \beta)^2 + 3 (\delta_p \beta)^3 + \dots) = E_t \hat{\pi}_{t+1}^p$$

- Simplify:

$$\begin{aligned} [\hat{p}_t^p - \delta_p \beta E_t \hat{p}_{t+1}^p] + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} [\hat{v}_t^p - \delta_p \beta E_t \hat{v}_{t+1}^p] &= E_t \hat{\pi}_{t+1}^p + ((1 - \delta_p \beta)^2) * \\ E_t \left[\frac{\widehat{m}c_{t+1} + \hat{\mu}_{t+1}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + (\delta_p \beta) \left(\frac{\widehat{m}c_{t+2} + \hat{\mu}_{t+2}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+2}^p \right) + (\delta_p \beta)^2 \left(\frac{\widehat{m}c_{t+3} + \hat{\mu}_{t+3}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p \right) + \dots \right] \end{aligned}$$

- Rewrite:

$$\begin{aligned} & [\hat{p}_t^p - \delta_p \beta E_t \hat{p}_{t+1}^p] + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} [\hat{v}_t^p - \delta_p \beta E_t \hat{v}_{t+1}^p] = E_t \hat{\pi}_{t+1}^p + ((1 - \delta_p \beta)^2) * \\ & \left[E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \frac{\widehat{m}c_{t+1+k} + \hat{\mu}_{t+1+k}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + E_t [(\delta_p \beta) \hat{\pi}_{t+2}^p + (\delta_p \beta)^2 (\hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p) + \dots] \right] \end{aligned}$$

- The final term can be rewritten using (29):

$$E_t [(\delta_p \beta) \hat{\pi}_{t+2}^p + (\delta_p \beta)^2 (\hat{\pi}_{t+2}^p + \hat{\pi}_{t+3}^p) + \dots] = \sum_{k=1}^{\infty} (\delta_p \beta)^k \hat{\Pi}_{t+1,k}^p$$

- Then we have:

$$\begin{aligned} & [\hat{p}_t^p - \delta_p \beta E_t \hat{p}_{t+1}^p] + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} [\hat{v}_t^p - \delta_p \beta E_t \hat{v}_{t+1}^p] = \\ & E_t \hat{\pi}_{t+1}^p + ((1 - \delta_p \beta)^2) E_t \left[\sum_{k=0}^{\infty} (\delta_p \beta)^k \frac{\widehat{m}c_{t+1+k} + \hat{\mu}_{t+1+k}^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \sum_{k=1}^{\infty} (\delta_p \beta)^k \hat{\Pi}_{t+1,k}^p \right] \end{aligned}$$

- Now we can replace the right-hand side using (28):

$$\begin{aligned} & [\hat{p}_t^p - \delta_p \beta E_t \hat{p}_{t+1}^p] + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} [\hat{v}_t^p - \delta_p \beta E_t \hat{v}_{t+1}^p] \\ & = E_t \hat{\pi}_{t+1}^p + (1 - \delta_p \beta) E_t \hat{p}_{t+1}^p + \delta_p \beta E_t \hat{v}_{t+1}^p \end{aligned}$$

- Further simplification:

$$\begin{aligned} & [\hat{p}_t^p - E_t \hat{p}_{t+1}^p + (1 - \delta_p \beta) E_t \hat{p}_{t+1}^p] + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} [\hat{v}_t^p - E_t \hat{v}_{t+1}^p + (1 - \delta_p \beta) E_t \hat{v}_{t+1}^p] \\ & = E_t \hat{\pi}_{t+1}^p + (1 - \delta_p \beta) E_t \hat{p}_{t+1}^p + \delta_p \beta E_t \hat{v}_{t+1}^p \end{aligned}$$

- Cancel terms:

$$\begin{aligned} & [\hat{p}_t^p - E_t \hat{p}_{t+1}^p] + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} [\hat{v}_t^p - E_t \hat{v}_{t+1}^p] \\ & = E_t \hat{\pi}_{t+1}^p - E_t \hat{v}_{t+1}^p \end{aligned}$$

- Equivalently:

$$\begin{aligned} & [E_t \hat{p}_{t+1}^p - \hat{p}_t^p] + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} [E_t \hat{v}_{t+1}^p - \hat{v}_t^p] \\ & = E_t \hat{v}_{t+1}^p - E_t \hat{\pi}_{t+1}^p \end{aligned}$$

- Equivalently:

$$E_t \hat{p}_{t+1}^p = \hat{p}_t^p + E_t \hat{v}_{t+1}^p - E_t \hat{\pi}_{t+1}^p - \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} [E_t \hat{v}_{t+1}^p - \hat{v}_t^p] \quad (31)$$

- Equivalently:

$$E_t \hat{p}_{t+1}^p - \hat{p}_t^p = \frac{-2\delta_p \beta}{(1 - \delta_p \beta)} E_t \hat{v}_{t+1}^p - E_t \hat{\pi}_{t+1}^p + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} \hat{v}_t^p \quad (32)$$

5.5.3 Combine (30) and (31)

- (30) for ease of reference:

$$\begin{aligned} & \delta_p \beta E_t \hat{p}_{t+1}^p - \hat{p}_t^p + \frac{\delta_p \beta}{(1 - \delta_p \beta)} [E_t \hat{v}_{t+1}^p - \hat{v}_t^p] \\ &= -(1 - \delta_p \beta) \left(\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \right) + \delta_p \beta E_t \hat{v}_{t+1}^p - \delta_p \beta E_t \hat{\pi}_{t+1}^p \end{aligned}$$

- Plug in (31):

$$\begin{aligned} & \delta_p \beta \left[\hat{p}_t^p + E_t \hat{v}_{t+1}^p - E_t \hat{\pi}_{t+1}^p - \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} [E_t \hat{v}_{t+1}^p - \hat{v}_t^p] \right] \\ & \quad - \hat{p}_t^p + \frac{\delta_p \beta}{(1 - \delta_p \beta)} [E_t \hat{v}_{t+1}^p - \hat{v}_t^p] \\ &= -(1 - \delta_p \beta) \left(\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \right) + \delta_p \beta E_t \hat{v}_{t+1}^p - \delta_p \beta E_t \hat{\pi}_{t+1}^p \end{aligned}$$

- Equivalently:

$$\begin{aligned} & (\delta_p \beta - 1) \hat{p}_t^p + \delta_p \beta E_t \hat{v}_{t+1}^p - \delta_p \beta E_t \hat{\pi}_{t+1}^p + \frac{\delta_p \beta}{(1 - \delta_p \beta)} (1 - 1 - \delta_p \beta) (E_t \hat{v}_{t+1}^p - \hat{v}_t^p) \\ &= -(1 - \delta_p \beta) \left(\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \right) + \delta_p \beta E_t \hat{v}_{t+1}^p - \delta_p \beta E_t \hat{\pi}_{t+1}^p \end{aligned}$$

- Cancel terms and multiply by -1 :

$$\begin{aligned} & (1 - \delta_p \beta) \hat{p}_t^p + \frac{(\delta_p \beta)^2}{(1 - \delta_p \beta)} (E_t \hat{v}_{t+1}^p - \hat{v}_t^p) \\ &= (1 - \delta_p \beta) \left(\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \right) \end{aligned}$$

- Simplify further:

$$\frac{(\delta_p \beta)^2}{(1 - \delta_p \beta)} (E_t \hat{v}_{t+1}^p - \hat{v}_t^p) = (1 - \delta_p \beta) \left(\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} - \hat{p}_t^p \right)$$

- Preliminary difference equation for \hat{v}_t^p :

$$(E_t \hat{v}_{t+1}^p - \hat{v}_t^p) = \frac{(1 - \delta_p \beta)^2}{(\delta_p \beta)^2} \left(\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} - \hat{p}_t^p \right) \quad (33)$$

6 THE PRICE INDEX

6.1 Formula for the Index

- Price Level:

$$P_t = \left[(1 - \delta_p) \sum_{s=0}^{\infty} \delta_p^s [V_{t-s}^p (v_{t-s}^p)^s]^{1-\sigma_t^p} \right]^{\frac{1}{1-\sigma_t^p}}$$

- Deflate by current target price level P_t^* and write out terms ($\check{P}_t = P_t/P_t^*$, $\check{V}_t^p = V_t^p/P_t^*$):

$$\begin{aligned} (\check{P}_t)^{1-\sigma_t^p} &= (1 - \delta_p) (\check{V}_t^p)^{1-\sigma_t^p} + (1 - \delta_p) \delta_p (\check{V}_{t-1}^p)^{1-\sigma_t^p} \left(\frac{v_{t-1}^p}{\pi_t^*} \right)^{1-\sigma_t^p} \\ &+ (1 - \delta_p) \delta_p^2 (\check{V}_{t-2}^p)^{1-\sigma_t^p} \left(\frac{(\hat{v}_{t-2}^{pp})^2}{\pi_t^* \pi_{t-1}^*} \right)^{(1-\sigma_t^p)} + (1 - \delta_p) \delta_p^3 (\check{V}_{t-3}^p)^{1-\sigma_t^p} \left(\frac{(v_{t-3}^p)^3}{\pi_t^* \pi_{t-1}^* \pi_{t-2}^*} \right)^{(1-\sigma_t^p)} + \dots \end{aligned}$$

- Divide by \check{P}_{t-1} :

$$\begin{aligned} \left(\frac{\check{P}_t}{\check{P}_{t-1}} \right)^{1-\sigma_t^p} &= (1 - \delta_p) \left(\frac{\check{V}_t^p}{\check{P}_t} \right)^{1-\sigma_t^p} \left(\frac{\check{P}_t}{\check{P}_{t-1}} \right)^{1-\sigma_t^p} \\ &+ (1 - \delta_p) \delta_p \left(\frac{\check{V}_{t-1}^p}{\check{P}_{t-1}} \right)^{1-\sigma_t^p} \left(\frac{v_{t-1}^p}{\pi_t^*} \right)^{1-\sigma_t^p} \\ &+ (1 - \delta_p) \delta_p^2 \left(\frac{\check{V}_{t-2}^p}{\check{P}_{t-2}} \right)^{1-\sigma_t^p} \left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}} \right)^{1-\sigma_t^p} \left(\frac{(v_{t-2}^p)^2}{\pi_t^* \pi_{t-1}^*} \right)^{(1-\sigma_t^p)} \\ &+ (1 - \delta_p) \delta_p^3 \left(\frac{\check{V}_{t-3}^p}{\check{P}_{t-3}} \right)^{1-\sigma_t^p} \left(\frac{\check{P}_{t-3}}{\check{P}_{t-2}} \right)^{1-\sigma_t^p} \left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}} \right)^{1-\sigma_t^p} \left(\frac{(v_{t-3}^p)^3}{\pi_t^* \pi_{t-1}^* \pi_{t-2}^*} \right)^{(1-\sigma_t^p)} + \dots \end{aligned}$$

- Use $\check{P}_t/\check{P}_{t-1} = \check{\pi}_t^p = \frac{\pi_t^p}{\pi_t^*}$ (this is the deviation of gross inflation from its target):

$$\begin{aligned}
(\check{\pi}_t^p)^{1-\sigma_t^p} &= (1 - \delta_p) (p_t)^{1-\sigma_t^p} (\check{\pi}_t^p)^{1-\sigma_t^p} \\
&+ (1 - \delta_p) \delta_p (p_{t-1})^{1-\sigma_t^p} \left(\frac{v_{t-1}^p}{\pi_t^*} \right)^{1-\sigma_t^p} \\
&+ (1 - \delta_p) \delta_p^2 (p_{t-2})^{1-\sigma_t^p} \left(\frac{1}{\check{\pi}_{t-1}^p} \right)^{1-\sigma_t^p} \left(\frac{(v_{t-2}^p)^p}{\pi_t^* \pi_{t-1}^*} \right)^{(1-\sigma_t^p)} \\
&+ (1 - \delta_p) \delta_p^3 (p_{t-3})^{1-\sigma_t^p} \left(\frac{1}{\check{\pi}_{t-2}^p} \right)^{1-\sigma_t^p} \left(\frac{1}{\check{\pi}_{t-1}^p} \right)^{1-\sigma_t^p} \left(\frac{(v_{t-3}^p)^3}{\pi_t^* \pi_{t-1}^* \pi_{t-2}^*} \right)^{(1-\sigma_t^p)} + \dots
\end{aligned}$$

- Divide through by $(\check{\pi}_t^p)^{1-\sigma_t^p}$:

$$\begin{aligned}
1 &= (1 - \delta_p) p_t^{1-\sigma_t^p} \\
&+ (1 - \delta_p) \delta_p p_{t-1}^{1-\sigma_t^p} \left(\frac{v_{t-1}^p}{\check{\pi}_t^p \pi_t^*} \right)^{1-\sigma_t^p} \\
&+ (1 - \delta_p) \delta_p^2 p_{t-2}^{1-\sigma_t^p} \left(\frac{(v_{t-2}^p)^2}{\check{\pi}_{t-1}^p \check{\pi}_t^p \pi_{t-1}^* \pi_t^*} \right)^{1-\sigma_t^p} \\
&+ (1 - \delta_p) \delta_p^3 p_{t-3}^{1-\sigma_t^p} \left(\frac{(v_{t-3}^p)^3}{\check{\pi}_{t-2}^p \check{\pi}_{t-1}^p \check{\pi}_t^p \pi_{t-2}^* \pi_{t-1}^* \pi_t^*} \right)^{1-\sigma_t^p} + \dots
\end{aligned}$$

- Let $\check{v}_{t-1}^p = v_{t-1}^p/\pi_{t-1}^*$, the deviation of firm-specific gross inflation from the inflation target, and use $\pi_t^*/\pi_{t-1}^* = \varepsilon_t^{\pi^*}$:

$$\begin{aligned}
1 &= (1 - \delta_p) p_t^{1-\sigma_t^p} \tag{34} \\
&+ (1 - \delta_p) \delta_p p_{t-1}^{1-\sigma_t^p} \left(\frac{\check{v}_{t-1}^p}{\check{\pi}_t^p \varepsilon_t^{\pi^*}} \right)^{1-\sigma_t^p} \\
&+ (1 - \delta_p) \delta_p^2 p_{t-2}^{1-\sigma_t^p} \left(\frac{(\check{v}_{t-2}^p)^2}{\check{\pi}_{t-1}^p \check{\pi}_t^p (\varepsilon_{t-1}^{\pi^*})^2 \varepsilon_t^{\pi^*}} \right)^{1-\sigma_t^p} \\
&+ (1 - \delta_p) \delta_p^3 p_{t-3}^{1-\sigma_t^p} \left(\frac{(\check{v}_{t-3}^p)^3}{\check{\pi}_{t-2}^p \check{\pi}_{t-1}^p \check{\pi}_t^p (\varepsilon_{t-2}^{\pi^*})^3 (\varepsilon_{t-1}^{\pi^*})^2 \varepsilon_t^{\pi^*}} \right)^{1-\sigma_t^p} + \dots
\end{aligned}$$

6.2 Linearize (34)

- Linearize:

$$\begin{aligned}
0 &= (1 - \delta_p)(1 - \sigma_t^p) \hat{p}_t^p \\
&\quad + (1 - \delta_p)(1 - \sigma_t^p) \delta_p (\hat{p}_{t-1}^p + \hat{v}_{t-1}^p - \hat{\pi}_t^p - \hat{\varepsilon}_t^{\pi^*}) \\
&\quad + (1 - \delta_p)(1 - \sigma_t^p) \delta_p^2 (\hat{p}_{t-2}^p + 2\hat{v}_{t-2}^p - \hat{\pi}_{t-1}^p - \hat{\pi}_t^p - 2\hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) \\
&\quad + (1 - \delta_p)(1 - \sigma_t^p) \delta_p^3 (\hat{p}_{t-3}^p + 3\hat{v}_{t-3}^p - \hat{\pi}_{t-2}^p - \hat{\pi}_{t-1}^p - \hat{\pi}_t^p - 3\hat{\varepsilon}_{t-2}^{\pi^*} - 2\hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) + \dots
\end{aligned}$$

- Cancel terms and bring $\hat{\pi}_t^p$ onto left-hand side:

$$\begin{aligned}
&\hat{\pi}_t^p (\delta_p + \delta_p^2 + \delta_p^3 + \dots) = \hat{p}_t^p + \delta_p (\hat{p}_{t-1}^p + \hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*}) \\
&+ \delta_p^2 (\hat{p}_{t-2}^p + 2\hat{v}_{t-2}^p - \hat{\pi}_{t-1}^p - 2\hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) + \delta_p^3 (\hat{p}_{t-3}^p + 3\hat{v}_{t-3}^p - \hat{\pi}_{t-2}^p - \hat{\pi}_{t-1}^p - 3\hat{\varepsilon}_{t-2}^{\pi^*} - 2\hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) + \dots
\end{aligned}$$

- Equivalently:

$$\begin{aligned}
\hat{\pi}_t^p &= \frac{1 - \delta_p}{\delta_p} \hat{p}_t^p + (1 - \delta_p) (\hat{p}_{t-1}^p + \hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*}) + (1 - \delta_p) \delta_p (\hat{p}_{t-2}^p + 2\hat{v}_{t-2}^p - \hat{\pi}_{t-1}^p - 2\hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) \\
&\quad + (1 - \delta_p) \delta_p^2 (\hat{p}_{t-3}^p + 3\hat{v}_{t-3}^p - \hat{\pi}_{t-2}^p - \hat{\pi}_{t-1}^p - 3\hat{\varepsilon}_{t-2}^{\pi^*} - 2\hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) + \dots
\end{aligned}$$

6.3 Quasi-Differencing

- Lag the last equation and multiply by δ_p :

$$\begin{aligned}
\delta_p \hat{\pi}_{t-1}^p &= (1 - \delta_p) \hat{p}_{t-1}^p + (1 - \delta_p) \delta_p (\hat{p}_{t-2}^p + \hat{v}_{t-2}^p - \hat{\varepsilon}_{t-1}^{\pi^*}) \\
&\quad + (1 - \delta_p) \delta_p^2 (\hat{p}_{t-3}^p + 2\hat{v}_{t-3}^p - \hat{\pi}_{t-2}^p - 2\hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*}) + \dots
\end{aligned}$$

- Deduct the last equation from the preceding one:

$$\begin{aligned}
\hat{\pi}_t^p - \delta_p \hat{\pi}_{t-1}^p &= \frac{1 - \delta_p}{\delta_p} \hat{p}_t^p + (1 - \delta_p) (\hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*}) + (1 - \delta_p) \delta_p (\hat{v}_{t-2}^p - \hat{\pi}_{t-1}^p - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) \\
&\quad + (1 - \delta_p) \delta_p^2 (\hat{v}_{t-3}^p - \hat{\pi}_{t-1}^p - \hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) + \dots
\end{aligned}$$

- Equivalently:

$$\begin{aligned}
\hat{\pi}_t^p - \delta_p \hat{\pi}_{t-1}^p &= \frac{1 - \delta_p}{\delta_p} \hat{p}_t^p - \delta_p \hat{\pi}_{t-1}^p + (1 - \delta_p) (\hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*}) + (1 - \delta_p) \delta_p (\hat{v}_{t-2}^p - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) \\
&\quad + (1 - \delta_p) \delta_p^2 (\hat{v}_{t-3}^p - \hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) + \dots
\end{aligned}$$

- Equivalently:

$$\begin{aligned}
\hat{\pi}_t^p &= \frac{1 - \delta_p}{\delta_p} \hat{p}_t^p + (1 - \delta_p) (\hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*}) + (1 - \delta_p) \delta_p (\hat{v}_{t-2}^p - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) \\
&\quad + (1 - \delta_p) \delta_p^2 (\hat{v}_{t-3}^p - \hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) + \dots
\end{aligned}$$

- We finally obtain the key expression discussed at some length in the paper:

$$\hat{\pi}_t^p = \frac{1 - \delta_p}{\delta_p} \hat{p}_t^p + \hat{\psi}_t^p \quad (35)$$

$$\hat{\psi}_t^p = (1 - \delta_p) (\hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*}) + (1 - \delta_p) \delta_p (\hat{v}_{t-2}^p - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) + (1 - \delta_p) \delta_p^2 (\hat{v}_{t-3}^p - \hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}) + \dots$$

6.4 The Auxiliary Variable $\hat{\psi}_t^p$

- Lag the last equation and multiply by δ_p :

$$\delta_p \hat{\psi}_{t-1}^p = (1 - \delta_p) \delta_p (\hat{v}_{t-2}^p - \hat{\varepsilon}_{t-1}^{\pi^*}) + (1 - \delta_p) \delta_p^2 (\hat{v}_{t-3}^p - \hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*}) + \dots$$

- Deduct the last equation from the preceding one:

$$\hat{\psi}_t^p = \delta_p \hat{\psi}_{t-1}^p + (1 - \delta_p) \hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*} \quad (36)$$

- Also, for future reference:

$$\left(E_t \hat{\psi}_{t+1}^p - \hat{\psi}_t^p \right) = (1 - \delta_p) \hat{v}_t^p - (1 - \delta_p) \hat{\psi}_t^p \quad (37)$$

- Finally we will also be able to use (35) to substitute \hat{p}_t^p out of equation (33):

$$\hat{p}_t^p = \frac{\delta_p}{1 - \delta_p} \left(\hat{\pi}_t^p - \hat{\psi}_t^p \right) \quad (38)$$

6.5 More Intuition for (36)

- Assume that we incorrectly linearize (34) around the period t inflation target π_t^* regardless of the time subscript of the variables. We get the following, after changing notation to distinguish this linearization from (36):

$$\widehat{\psi}_t = \delta_p \widehat{\psi}_{t-1} + (1 - \delta_p) \widehat{v}_{t-1}^p \quad (39)$$

- Note that for \widehat{v}_{t-1}^p we have (the same holds for $\widehat{\psi}_{t-1}$)

$$\begin{aligned} \widehat{v}_{t-1}^p &= \ln(v_{t-1}^p) - \ln(\pi_t^*) \quad , \quad \text{while} \\ \widehat{v}_{t-1}^p &= \ln(v_{t-1}^p) - \ln(\pi_{t-1}^*) \end{aligned}$$

- This implies that (again similarly for $\widehat{\psi}_{t-1}$)

$$\widehat{v}_{t-1}^p = \hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*}$$

- Also

$$\widehat{\psi}_t = \hat{\psi}_t^p$$

- Substituting the foregoing into (39) we obtain (36):

$$\hat{\psi}_t^p = \delta_p \hat{\psi}_{t-1}^p + (1 - \delta_p) \hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*}$$

7 FINAL INFLATION DYNAMICS

7.1 Final Equation for \hat{v}_t^p

- Equation (33) reproduced for ease of reference:

$$(E_t \hat{v}_{t+1}^p - \hat{v}_t^p) = \frac{(1 - \delta_p \beta)^2}{(\delta_p \beta)^2} \left(\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} - \hat{p}_t^p \right)$$

- Plug (38) into this:

$$E_t \hat{v}_{t+1}^p = \hat{v}_t^p + \frac{(1 - \delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\delta_p}{1 - \delta_p} \hat{\psi}_t^p - \frac{(1 - \delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\delta_p}{1 - \delta_p} \hat{\pi}_t^p + \frac{(1 - \delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \quad (40)$$

7.2 Final Equation for $\hat{\pi}_t^p$

- Deduct (35) from its once led version to get:

$$(E_t \hat{\pi}_{t+1}^p - \hat{\pi}_t^p) = \frac{1 - \delta_p}{\delta_p} (E_t \hat{p}_{t+1}^p - \hat{p}_t^p) + (E_t \hat{\psi}_{t+1}^p - \hat{\psi}_t^p)$$

- To substitute for the first term on the right-hand side, use (32), which we reproduce here:

$$E_t \hat{p}_{t+1}^p - \hat{p}_t^p = \frac{-2\delta_p \beta}{(1 - \delta_p \beta)} E_t \hat{v}_{t+1}^p - E_t \hat{\pi}_{t+1}^p + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} \hat{v}_t^p$$

- To substitute for the second term on the right-hand side, use (37), which we reproduce here:

$$(E_t \hat{\psi}_{t+1}^p - \hat{\psi}_t^p) = (1 - \delta_p) \hat{v}_t^p - (1 - \delta_p) \hat{\psi}_t^p$$

- Substitute:

$$(E_t \hat{\pi}_{t+1}^p - \hat{\pi}_t^p) = (1 - \delta_p) \hat{v}_t^p - (1 - \delta_p) \hat{\psi}_t^p + \frac{1 - \delta_p}{\delta_p} \left[\frac{-2\delta_p \beta}{(1 - \delta_p \beta)} E_t \hat{v}_{t+1}^p - E_t \hat{\pi}_{t+1}^p + \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} \hat{v}_t^p \right]$$

- Equivalently:

$$E_t \hat{\pi}_{t+1}^p \left(1 + \frac{1 - \delta_p}{\delta_p} \right) = \hat{\pi}_t^p - \frac{1 - \delta_p}{\delta_p} \frac{2\delta_p \beta}{(1 - \delta_p \beta)} E_t \hat{v}_{t+1}^p + \left(\frac{1 - \delta_p}{\delta_p} \frac{(1 + \delta_p \beta)}{(1 - \delta_p \beta)} + (1 - \delta_p) \right) \hat{v}_t^p - (1 - \delta_p) \hat{\psi}_t^p$$

- Simplify and substitute from (40) for $E_t \hat{v}_{t+1}^p$:

$$E_t \hat{\pi}_{t+1}^p \frac{1}{\delta_p} = \hat{\pi}_t^p + \left(\frac{1 - \delta_p (1 + \delta_p \beta)}{\delta_p (1 - \delta_p \beta)} + (1 - \delta_p) \right) \hat{v}_t^p - (1 - \delta_p) \hat{\psi}_t^p$$

$$- \frac{1 - \delta_p}{\delta_p} \frac{2\delta_p \beta}{(1 - \delta_p \beta)} \left[\hat{v}_t^p + \frac{(1 - \delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\delta_p}{1 - \delta_p} \hat{\psi}_t^p - \frac{(1 - \delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\delta_p}{1 - \delta_p} \hat{\pi}_t^p + \frac{(1 - \delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \right]$$

- Equivalently:

$$E_t \hat{\pi}_{t+1}^p \frac{1}{\delta_p} = \hat{\pi}_t^p \left(1 + \frac{2(1 - \delta_p \beta)}{(\delta_p \beta)} \right) + \hat{v}_t^p \left(\frac{1 - \delta_p (1 + \delta_p \beta)}{\delta_p (1 - \delta_p \beta)} + (1 - \delta_p) - \frac{1 - \delta_p}{\delta_p} \frac{2\delta_p \beta}{(1 - \delta_p \beta)} \right)$$

$$- \hat{\psi}_t^p \left(\frac{2(1 - \delta_p \beta)}{(\delta_p \beta)} + (1 - \delta_p) \right) - \frac{1 - \delta_p}{\delta_p} \frac{2(1 - \delta_p \beta)}{(\delta_p \beta)} \frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)}$$

- Equivalently:

$$E_t \hat{\pi}_{t+1}^p \frac{1}{\delta_p} = \hat{\pi}_t^p \left(\frac{\delta_p \beta + 2 - 2\delta_p \beta}{\delta_p \beta} \right) + \hat{v}_t^p \left(\frac{1 - \delta_p (1 + \delta_p \beta - 2\delta_p \beta)}{\delta_p (1 - \delta_p \beta)} + (1 - \delta_p) \right)$$

$$- \hat{\psi}_t^p \left(\frac{2 - 2\delta_p \beta + \delta_p \beta - \delta_p^2 \beta}{\delta_p \beta} \right) - \frac{1 - \delta_p}{\delta_p} \frac{2(1 - \delta_p \beta)}{(\delta_p \beta)} \frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)}$$

- Equivalently:

$$E_t \hat{\pi}_{t+1}^p \frac{1}{\delta_p} = \hat{\pi}_t^p \left(\frac{2 - \delta_p \beta}{\delta_p \beta} \right) + \hat{v}_t^p \left(\frac{(1 - \delta_p)(1 + \delta_p)}{\delta_p} \right)$$

$$- \hat{\psi}_t^p \left(\frac{2 - \delta_p \beta - \delta_p^2 \beta}{\delta_p \beta} \right) - \frac{1 - \delta_p}{\delta_p} \frac{2(1 - \delta_p \beta)}{(\delta_p \beta)} \frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)}$$

- Equivalently:

$$E_t \hat{\pi}_{t+1}^p = \hat{\pi}_t^p \left(\frac{2}{\beta} - \delta_p \right) + \hat{v}_t^p ((1 - \delta_p)(1 + \delta_p)) \tag{41}$$

$$+ \hat{\psi}_t^p \left(\delta_p (1 + \delta_p) - \frac{2}{\beta} \right) - \frac{2(1 - \delta_p)(1 - \delta_p \beta)}{(\delta_p \beta)(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} (\widehat{m}c_t + \hat{\mu}_t^p)$$

8 THE CALVO-YUN CASE

8.1 Profit Maximization

- Pricing policy of firm j that reoptimizes at t , choosing V_t^p and indexing to π_t^* (the current inflation target):

$$P_{t+k}(j) = V_t^p \Pi_{t,k}^*$$

- Define additional terms:
 - Cumulative inflation targets:

$$\Pi_{t,k}^* \equiv \prod_{j=1}^k \pi_{t+j}^* \text{ for } k \geq 1 \quad (\equiv 1 \text{ for } k = 0)$$

- Cumulative lagged aggregate inflation deviation:

$$\hat{\Pi}_{t,k}^* \equiv \sum_{j=1}^k \hat{\pi}_{t+j}^* \text{ for } k \geq 1 \quad (\equiv 0 \text{ for } k = 0)$$

- Profit maximization:

$$\underset{V_t^p}{Max} E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \lambda_{t+k} \left[\frac{V_t^p \Pi_{t,k}^*}{P_t \Pi_{t,k}^*} y_{t+k}(j) - mc_{t+k} y_{t+k}(j) - \frac{\phi_p}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}} \right], \text{ s.t.}$$

$$y_{t+k}(j) = Y_{t+k} \left(\frac{V_t^p \Pi_{t,k}^*}{P_t \Pi_{t,k}^*} \right)^{-\sigma_{t+k}^p}$$

- Substitute constraints:

$$\underset{V_t^p, v_t^p}{Max} E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \lambda_{t+k} \left[\left(\frac{V_t^p \Pi_{t,k}^*}{P_t \Pi_{t,k}^*} \right)^{1-\sigma_{t+k}^p} Y_{t+k} - mc_{t+k} \left(\frac{V_t^p \Pi_{t,k}^*}{P_t \Pi_{t,k}^*} \right)^{-\sigma_{t+k}^p} Y_{t+k} - \frac{\phi_p}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}} \right]$$

- FOC for V_t^p , normalized by inflation target:

$$p_t = \frac{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k}^p \left(mc_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right)}{E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) (\sigma_{t+k}^p - 1) \left(\frac{1}{\hat{\Pi}_{t,k}^*} \right)} \quad (42)$$

- Linearization of goods demand:

$$\left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k} \right) = -\bar{\sigma}_p \left(\hat{p}_t^p - \hat{\Pi}_{t,k}^p \right)$$

- Rewrite (42):

$$\begin{aligned}
& p_t E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) (\sigma_{t+k}^p - 1) \left(\frac{1}{\check{\Pi}_{t,k}} \right) \\
&= E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k}^p \left(mc_{t+k} + \phi_p \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right)
\end{aligned}$$

- Linearization of (42):

$$\begin{aligned}
& \frac{\hat{p}_t^p}{1 - \delta_p \beta} - E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \hat{\Pi}_{t,k}^p \\
&= E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \left[\widehat{mc}_{t+k} + \hat{\mu}_{t+k}^p + \phi_p \bar{\mu}_p \left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k} \right) \right]
\end{aligned}$$

- Equivalently:

$$\frac{\hat{p}_t^p}{1 - \delta_p \beta} - E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \hat{\Pi}_{t,k}^p = E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \frac{(\widehat{mc}_{t+k} + \hat{\mu}_{t+k}^p)}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)}$$

- Equivalently:

$$\frac{\hat{p}_t^p}{1 - \delta_p \beta} = E_t \sum_{k=0}^{\infty} (\delta_p \beta)^k \left[\frac{(\widehat{mc}_{t+k} + \hat{\mu}_{t+k}^p)}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} + \hat{\Pi}_{t,k}^p \right]$$

- Quasi-Differencing:

$$\hat{p}_t^p - \delta_p \beta E_t \hat{p}_{t+1}^p = (1 - \delta_p \beta) \left[\frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \right] + \delta_p \beta E_t \hat{\pi}_{t+1}^p \quad (43)$$

8.2 The Price Index

- Price Level:

$$P_t = \left[(1 - \delta_p) \sum_{s=0}^{\infty} \delta_p^s [V_{t-s}^p \Pi_{t-s,s}^*]^{1-\sigma_t^p} \right]^{\frac{1}{1-\sigma_t^p}}$$

- Deflate by current target price level P_t^* and write out terms ($\check{P}_t = P_t/P_t^*$, $\check{V}_t^p = V_t^p/P_t^*$):

$$(\check{P}_t)^{1-\sigma_t^p} = (1-\delta_p) (\check{V}_t^p)^{1-\sigma_t^p} + (1-\delta_p) \delta_p (\check{V}_{t-1}^p)^{1-\sigma_t^p} + (1-\delta_p) \delta_p^2 (\check{V}_{t-2}^p)^{1-\sigma_t^p} + (1-\delta_p) \delta_p^3 (\check{V}_{t-3}^p)^{1-\sigma_t^p} + \dots$$

- Divide by \check{P}_{t-1} :

$$\begin{aligned}
\left(\frac{\check{P}_t}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} &= (1-\delta_p) \left(\frac{\check{V}_t^p}{\check{P}_t}\right)^{1-\sigma_t^p} \left(\frac{\check{P}_t}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} \\
&\quad + (1-\delta_p)\delta_p \left(\frac{\check{V}_{t-1}^p}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} \\
&\quad + (1-\delta_p)\delta_p^2 \left(\frac{\check{V}_{t-2}^p}{\check{P}_{t-2}}\right)^{1-\sigma_t^p} \left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} \\
&\quad + (1-\delta_p)\delta_p^3 \left(\frac{\check{V}_{t-3}^p}{\check{P}_{t-3}}\right)^{1-\sigma_t^p} \left(\frac{\check{P}_{t-3}}{\check{P}_{t-2}}\right)^{1-\sigma_t^p} \left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}}\right)^{1-\sigma_t^p} + \dots
\end{aligned}$$

- Use $\check{P}_t/\check{P}_{t-1} = \check{\pi}_t^p = \frac{\pi_t^p}{\pi_t^*}$:

$$\begin{aligned}
(\check{\pi}_t^p)^{1-\sigma_t^p} &= (1-\delta_p) (p_t)^{1-\sigma_t^p} (\check{\pi}_t^p)^{1-\sigma_t^p} \\
&\quad + (1-\delta_p)\delta_p (p_{t-1})^{1-\sigma_t^p} \\
&\quad + (1-\delta_p)\delta_p^2 (p_{t-2})^{1-\sigma_t^p} \left(\frac{1}{\check{\pi}_{t-1}^p}\right)^{1-\sigma_t^p} \\
&\quad + (1-\delta_p)\delta_p^3 (p_{t-3})^{1-\sigma_t^p} \left(\frac{1}{\check{\pi}_{t-2}^p}\right)^{1-\sigma_t^p} \left(\frac{1}{\check{\pi}_{t-1}^p}\right)^{1-\sigma_t^p} + \dots
\end{aligned}$$

- Divide through by $(\check{\pi}_t^p)^{1-\sigma_t^p}$:

$$\begin{aligned}
1 &= (1-\delta_p)p_t^{1-\sigma_t^p} \\
&\quad + (1-\delta_p)\delta_p p_{t-1}^{1-\sigma_t^p} \left(\frac{1}{\check{\pi}_t^p}\right)^{1-\sigma_t^p} \\
&\quad + (1-\delta_p)\delta_p^2 p_{t-2}^{1-\sigma_t^p} \left(\frac{1}{\check{\pi}_{t-1}^p \check{\pi}_t^p}\right)^{1-\sigma_t^p} \\
&\quad + (1-\delta_p)\delta_p^3 p_{t-3}^{1-\sigma_t^p} \left(\frac{1}{\check{\pi}_{t-2}^p \check{\pi}_{t-1}^p \check{\pi}_t^p}\right)^{1-\sigma_t^p} + \dots
\end{aligned}$$

- Linearize:

$$0 = \hat{p}_t^p + \delta_p (\hat{p}_{t-1}^p - \hat{\pi}_t^p) + \delta_p^2 (\hat{p}_{t-2}^p - \hat{\pi}_{t-1}^p - \hat{\pi}_t^p) + \delta_p^3 (\hat{p}_{t-3}^p - \hat{\pi}_{t-2}^p - \hat{\pi}_{t-1}^p - \hat{\pi}_t^p) + \dots$$

- Cancel terms and bring $\hat{\pi}_t^p$ onto left-hand side:

$$\hat{\pi}_t^p (\delta_p + \delta_p^2 + \delta_p^3 + \dots) = \hat{p}_t^p + \delta_p \hat{p}_{t-1}^p + \delta_p^2 (\hat{p}_{t-2}^p - \hat{\pi}_{t-1}^p) + \delta_p^3 (\hat{p}_{t-3}^p - \hat{\pi}_{t-2}^p - \hat{\pi}_{t-1}^p) + \dots$$

- Equivalently:

$$\hat{\pi}_t^p = \frac{1 - \delta_p}{\delta_p} \hat{p}_t^p + (1 - \delta_p) \hat{p}_{t-1}^p + (1 - \delta_p) \delta_p (\hat{p}_{t-2}^p - \hat{\pi}_{t-1}^p) + (1 - \delta_p) \delta_p^2 (\hat{p}_{t-3}^p - \hat{\pi}_{t-2}^p - \hat{\pi}_{t-1}^p) + \dots$$

- Lag the last equation and multiply by δ_p :

$$\delta_p \hat{\pi}_{t-1}^p = (1 - \delta_p) \hat{p}_{t-1}^p + (1 - \delta_p) \delta_p \hat{p}_{t-2}^p + (1 - \delta_p) \delta_p^2 (\hat{p}_{t-3}^p - \hat{\pi}_{t-2}^p) +$$

- Deduct the last equation from the preceding one:

$$\hat{\pi}_t^p = \frac{1 - \delta_p}{\delta_p} \hat{p}_t^p \quad (44)$$

8.3 Final Inflation Dynamics

- Plug (44) into (43):

$$\frac{\delta_p}{1 - \delta_p} \hat{\pi}_t^p - \delta_p \beta \frac{\delta_p}{1 - \delta_p} E_t \hat{\pi}_{t+1}^p = (1 - \delta_p \beta) \left[\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \right] + \delta_p \beta E_t \hat{\pi}_{t+1}^p$$

- Simplify:

$$\hat{\pi}_t^p = \beta E_t \hat{\pi}_{t+1}^p + \frac{(1 - \delta_p \beta)(1 - \delta_p)}{\delta_p} \left[\frac{\widehat{m}c_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \right] \quad (45)$$

- This is the New Keynesian Phillips Curve for the Calvo-Yun case, which replaces the three equation pricing block of the optimal indexation system.

9 HOUSEHOLD WAGE SETTING

9.1 Utility Maximization

- Wage setting policy of a worker i that reoptimizes at t , choosing V_t^w and v_t^w (a gross wage inflation rate):

$$W_{t+k}(i) = V_t^w (v_t^w)^k$$

- Define terms:

- Front-loading term:

$$p_t^w \equiv \frac{V_t^w}{W_t}$$

- Real wage:

$$w_t = \frac{W_t}{P_t}$$

- Wage inflation rates re-scaled by the inflation target, with $\tilde{\pi}_t^w = \pi_t^w / \pi_t^*$.
- Cumulative aggregate rescaled wage inflation:

$$\tilde{\Pi}_{t,k}^w = \prod_{j=1}^k \tilde{\pi}_{t+j}^w \text{ for } k \geq 1 \quad (\equiv 1 \text{ for } k = 0)$$

- Cumulative aggregate rescaled wage inflation deviation:

$$\hat{\Pi}_{t,k}^w = \sum_{j=1}^k \hat{\pi}_{t+j}^w \text{ for } k \geq 1 \quad (\equiv 0 \text{ for } k = 0)$$

- Utility maximization - relevant part of the problem:

$$\begin{aligned} \underset{V_t^w, v_t^w}{Max} \quad & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left[-\psi S_t^L \frac{(L_{t+k}(i))^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \lambda_{t+k} \frac{V_t^w (v_t^w)^k}{W_{t+k}} \frac{W_{t+k}}{P_{t+k}} L_{t+k}(i) \right. \\ & \left. - \lambda_{t+k} \frac{\phi_w}{2} \frac{W_{t+k}}{P_{t+k}} \frac{(L_{t+k}(i) - \ell_{t+k})^2}{\ell_{t+k}} \right], \text{ s.t.} \\ & L_{t+k}(i) = \ell_{t+k} \left(\frac{V_t^w (v_t^w)^k}{W_{t+k}} \right)^{-\sigma_t^w} \end{aligned}$$

9.2 First-Order Conditions

- Real wage rescaled by technology, with $\check{w}_t = w_t/S_t^y$.
- Firm specific wage inflation rescaled by the inflation target $\check{v}_t^w = v_t^w/\pi_t^*$.
 - By the unit root property we can ignore future changes in the inflation target (see discussion above for firms):

$$E_t \hat{v}_{t+k}^w = E_t (\ln(v_{t+k}^w) - \ln(\pi_{t+k}^*)) = E_t (\ln(v_{t+k}^w) - \ln(\pi_t^*))$$

- FOC for V_t^w :

$$\begin{aligned} & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \check{\lambda}_{t+k} L_{t+k}(i) \left(\check{w}_{t+k} (\sigma_{t+k}^w - 1) p_t^w \frac{(\check{v}_t^w)^k}{\check{\Pi}_{t,k}^w} - \phi_w \check{w}_{t+k} \sigma_{t+k}^w \left(\frac{L_{t+k}(i) - \ell_{t+k}}{\ell_{t+k}} \right) \right) \\ = & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \psi \sigma_{t+k}^w S_{t+k}^L (L_{t+k}(i))^{1+\frac{1}{\eta}} \end{aligned} \quad (46)$$

- FOC w.r.t. v_t^w :

$$\begin{aligned} & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \check{\lambda}_{t+k} L_{t+k}(i) \left(\check{w}_{t+k} (\sigma_{t+k}^w - 1) p_t^w \frac{(\check{v}_t^w)^k}{\check{\Pi}_{t,k}^w} - \phi_w \check{w}_{t+k} \sigma_{t+k}^w \left(\frac{L_{t+k}(i) - \ell_{t+k}}{\ell_{t+k}} \right) \right) \\ = & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \psi \sigma_{t+k}^w S_{t+k}^L (L_{t+k}(i))^{1+\frac{1}{\eta}} \end{aligned} \quad (47)$$

9.3 Linearization

9.3.1 Linearization for V_t^w

- First step:

$$\begin{aligned} & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left[\hat{\lambda}_{t+k} + \hat{L}_{t+k}(i) + \hat{w}_{t+k} + \left(\hat{p}_t^w + k \hat{v}_t^w - \hat{\Pi}_{t,k}^w \right) \right] \\ &= E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left[\hat{\mu}_{t+k}^w + \hat{S}_{t+k}^L + \left(1 + \frac{1}{\eta} \right) \hat{L}_{t+k}(i) + \phi_w \bar{\mu}_w \left(\hat{L}_{t+k}(i) - \hat{\ell}_{t+k} \right) \right] \end{aligned}$$

- Linearization of labor demand:

$$\left(\hat{L}_{t+k}(i) - \hat{\ell}_{t+k} \right) = -\bar{\sigma}_w \left(\hat{p}_t^w + k \hat{v}_t^w - \hat{\Pi}_{t,k}^w \right)$$

- Marginal rate of substitution:

- MRS in levels:

$$mrs_t = \frac{S_t^L \psi L_t(i)^{\frac{1}{\eta}}}{\lambda_t}$$

- Log-linearized:

$$\widehat{mrs}_t = \frac{1}{\eta} \hat{L}_t(i) + \hat{S}_t^L - \hat{\lambda}_t$$

- Combine with the expression for labor demand (note that, for contemporaneous terms, $k = 0$):

$$\widehat{mrs}_t = \frac{1}{\eta} \hat{\ell}_t - \frac{\bar{\sigma}_w}{\eta} \hat{p}_t^w + \hat{S}_t^L - \hat{\lambda}_t$$

- Combine the above:

$$E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left(\hat{p}_t^w + k \hat{v}_t^w - \hat{\Pi}_{t,k}^w \right) = E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \frac{\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w \right)}{\left(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w \right)}$$

- Apply formulas (21) and (22):

$$\frac{\hat{p}_t^w}{1 - \delta_w \beta} + \frac{\hat{v}_t^w \delta_w \beta}{(1 - \delta_w \beta)^2} = E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left[\frac{\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w \right)}{\left(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w \right)} + \hat{\Pi}_{t,k}^w \right] \quad (48)$$

- With the appropriate change in notation this is exactly identical to equation (25) for price setting, after replacing $\left(\widehat{mcs}_{t+k} + \hat{\mu}_{t+k} \right) / \left(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p \right)$ with $\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w \right) / \left(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w \right)$.

9.3.2 Linearization for v_t^w

- First step:

$$\begin{aligned} & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left[\hat{\lambda}_{t+k} + \hat{L}_{t+k}(i) + \hat{w}_{t+k} + \left(\hat{p}_t^w + k \hat{v}_t^w - \hat{\Pi}_{t,k}^w \right) \right] \\ &= E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left[\hat{\mu}_{t+k}^w + \hat{S}_{t+k}^L + \left(1 + \frac{1}{\eta} \right) \hat{L}_{t+k}(i) + \phi_w \bar{\mu}_w \left(\hat{L}_{t+k}(i) - \hat{\ell}_{t+k} \right) \right] \end{aligned}$$

- Combine with the above:

$$E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left(\hat{p}_t^w + k \hat{v}_t^w - \hat{\Pi}_{t,k}^w \right) = E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \frac{(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)}$$

- Apply formulas (21) and (22):

$$\begin{aligned} & \frac{\hat{p}_t^w \delta_w \beta}{(1 - \delta_w \beta)^2} + \frac{\hat{v}_t^w \delta_w \beta (1 + \delta_w \beta)}{(1 - \delta_w \beta)^3} \tag{49} \\ &= E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left[\frac{(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)} + \hat{\Pi}_{t,k}^w \right] \end{aligned}$$

- With the appropriate change in notation this is exactly identical to equation (26) for price setting, after replacing $(\widehat{mc}_{t+k} + \hat{\mu}_{t+k}^p) / (1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)$ with $(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w) / (1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)$.

9.4 Final Wage Inflation Dynamics

- Given the identical forms of (48)/(25) and (49)/(26), we obtain after quasi-differencing as the end result an equation analogous to (33):

$$(E_t \hat{v}_{t+1}^w - \hat{v}_t^w) = \frac{(1 - \delta_w \beta)^2}{(\delta_w \beta)^2} \left(\frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}_t^w)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)} - \hat{p}_t^w \right) \quad (50)$$

- Furthermore, the derivation of the wage index is identical to that for the price index above. The correct expression for wage inflation is:

$$\hat{\pi}_t^w = \hat{w}_t - \hat{w}_{t-1} + \hat{g}_t - \hat{\pi}_t^p \quad (51)$$

- Furthermore, the derivation of the wage index is identical to that for the price index above. We get:

$$\hat{\pi}_t^w = \frac{1 - \delta_w}{\delta_w} \hat{p}_t^w + \hat{\psi}_t^w \quad (52)$$

$$\hat{\psi}_t^w = \delta_w \hat{\psi}_{t-1}^w + (1 - \delta_w) \hat{v}_{t-1}^w - \hat{\varepsilon}_t^{\pi^*} \quad (53)$$

- Combining this with the results of quasi-differencing we obtain the equations for $E_t \hat{v}_{t+1}^w$ and $E_t \hat{\pi}_{t+1}^w$:

$$\begin{aligned} E_t \hat{v}_{t+1}^w &= \hat{v}_t^w + \frac{(1 - \delta_w \beta)^2}{(\delta_w \beta)^2} \frac{\delta_w}{1 - \delta_w} \hat{\psi}_t^w - \frac{(1 - \delta_w \beta)^2}{(\delta_w \beta)^2} \frac{\delta_w}{1 - \delta_w} \hat{\pi}_t^w \\ &\quad + \frac{(1 - \delta_w \beta)^2}{(\delta_w \beta)^2} \frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}_t^w)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)} \end{aligned} \quad (54)$$

$$\begin{aligned} E_t \hat{\pi}_{t+1}^w &= \hat{\pi}_t^w \left(\frac{2}{\beta} - \delta_w \right) + \hat{v}_t^w ((1 - \delta_w) (1 + \delta_w)) \\ &\quad + \hat{\psi}_t^w \left(\delta_w (1 + \delta_w) - \frac{2}{\beta} \right) - \frac{2(1 - \delta_w) (1 - \delta_w \beta)}{(\delta_w \beta)} \frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}_t^w)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)} \end{aligned} \quad (55)$$

- We can also use (52) to get a final expression for the marginal rate of substitution:

$$\widehat{mrs}_t = \frac{1}{\eta} \hat{\ell}_t - \frac{\bar{\sigma}^w}{\eta} \frac{\delta_w}{1 - \delta_w} \hat{\pi}_t^w + \frac{\bar{\sigma}^w}{\eta} \frac{\delta_w}{1 - \delta_w} \hat{\psi}_t^w + \hat{S}_t^L - \hat{\lambda}_t \quad (56)$$

9.5 The Calvo-Yun Case

By analogy with the derivation under price setting we obtain the following for wage setting:

$$\hat{\pi}_t^w = \beta E_t \hat{\pi}_{t+1}^w + \frac{(1 - \delta_w \beta)(1 - \delta_w)}{\delta_w} \left(\frac{\widehat{mrs}_t - \hat{w}_t + \hat{\mu}_t^w}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)} \right) \quad (57)$$

10 GOVERNMENT AND MARKET CLEARING

- Interest rate rule in levels:
 - Re-scaled by dividing through by π_t^* .
 - Remember that $\check{i}_t = i_t/\pi_t^*$.
 - Let $\bar{r} = \bar{g}/\beta$.

$$\left(\frac{i_t}{\pi_t^*}\right)^4 = \left[\left(\frac{i_{t-1}}{\pi_{t-1}^*} \frac{\pi_{t-1}^*}{\pi_t^*}\right)^4\right]^{\theta^{int}} \left[\bar{r}^{-4} \frac{\pi_{t+3}^p \pi_{t+2}^p \pi_{t+1}^p \pi_t^p}{(\pi_t^*)^4}\right]^{1-\theta^{int}+\theta^\pi} \left[\frac{\check{Y}_{t+3}}{\check{Y}_{t-1}}\right]^{\theta^y}$$

- Equivalently:

$$(\check{i}_t)^4 = \left[\left(\frac{\check{i}_{t-1}}{\varepsilon_t^{\pi^*}}\right)^4\right]^{\theta^{int}} \left[\bar{r}^{-4} \check{\pi}_{t+3}^p \check{\pi}_{t+2}^p \check{\pi}_{t+1}^p \check{\pi}_t^p\right]^{1-\theta^{int}+\theta^\pi} \left[\frac{\check{Y}_{t+3}}{\check{Y}_{t-1}}\right]^{\theta^y}$$

- Linearized interest rate rule:

$$\begin{aligned} \hat{i}_t = & \theta^{int} (\hat{i}_{t-1} - \hat{\varepsilon}_t^{\pi^*}) + (1 - \theta^{int} + \theta^\pi) E_t \left(\frac{\hat{\pi}_{t+3}^p + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+1}^p + \hat{\pi}_t^p}{4} \right) \\ & + \theta^y E_t \left(\frac{\hat{Y}_{t+3} - \hat{Y}_{t-1}}{4} \right) + \frac{\hat{S}_t^{int}}{4} \end{aligned} \quad (58)$$

- Real Interest Rate:

$$\hat{r}_t = \hat{i}_t - E_t \hat{\pi}_{t+1}^p \quad (59)$$

- Steady State Spending = Fixed Fraction of GDP:

$$\begin{aligned} \overline{GOV} &= s_g \bar{Y} \\ G\check{O}V_t &= S_t^{gov} \overline{GOV} \end{aligned}$$

- Goods Market Clearing:

- In levels:

$$Y_t = C_t + I_t + GOV_t$$

- Rescaled by technology:

$$\check{Y}_t = \check{C}_t + \check{I}_t + G\check{O}V_t$$

- Linearization:

$$\bar{Y} \hat{Y}_t = \bar{C} \hat{C}_t + \bar{I} \hat{I}_t + \overline{GOV} \hat{S}_t^g \quad (60)$$

11 STATIONARY EXOGENOUS SHOCKS

- Firm Risk:

$$\hat{S}_t^\sigma = \rho^\sigma \hat{S}_{t-1}^\sigma + \hat{\varepsilon}_t^\sigma \quad (61)$$

- Consumption:

$$\hat{S}_t^c = \rho^c \hat{S}_{t-1}^c + \hat{\varepsilon}_t^c \quad (62)$$

- Labor Supply:

$$\hat{S}_t^L = \rho^L \hat{S}_{t-1}^L + \hat{\varepsilon}_t^L \quad (63)$$

- Investment Demand:

$$\hat{S}_t^{inv} = \rho^{inv} \hat{S}_{t-1}^{inv} + \hat{\varepsilon}_t^{inv} \quad (64)$$

- Government Spending:

$$\hat{S}_t^{gov} = \rho^{gov} \hat{S}_{t-1}^{gov} + \hat{\varepsilon}_t^{gov} \quad (65)$$

- Monetary Policy:

$$\hat{S}_t^{int} = \rho^{int} \hat{S}_{t-1}^{int} + \hat{\varepsilon}_t^{int} \quad (66)$$

- Goods Mark-Up:

$$\hat{\mu}_t^p = \hat{\varepsilon}_t^{\mu^p} \quad (67)$$

- Labor Mark-Up:

$$\hat{\mu}_t^w = \hat{\varepsilon}_t^{\mu^w} \quad (68)$$

12 COMPLETE DYNAMIC SYSTEM

12.1 Steady State and Parameter Calibration

12.1.1 Calibrated Directly

$$v = 0.7 \text{ (Habit Persistence)}$$

$$\eta = 0.5 \text{ (Labor Supply Elasticity)}$$

$$\phi_I = 0.7 \text{ (Investment Adjustment Costs)}$$

$$\sigma_a = 15 \text{ (Capacity Utilization Elasticity)}$$

$$\delta_p = 0.75 \text{ (Calvo delta for Prices)}$$

$$\delta_w = 0.75 \text{ (Calvo delta for Wages)}$$

$$\bar{\mu}_p = 1.23 \text{ (Price Markup)} \quad (\text{SS2})$$

$$\bar{\mu}_w = 1.16 \text{ (Wage Markup)} \quad (\text{SS2})$$

$$\phi_p = 0.75 \text{ (Deviation Cost for Prices)}$$

$$\phi_w = 0.75 \text{ (Deviation Cost for Wages)}$$

$$\theta^{int} = 0.75 \text{ (Interest Rate Smoothing)}$$

$$\theta^\pi = 0.25 \text{ (One minus Inflation Coefficient)}$$

$$\bar{L} = 1/3 \text{ (Labor Supply, } \psi \text{ is endogenous to that choice)} \quad (\text{SS1})$$

$$s_L = 0.64 \text{ (Labor Share - after adjusting for markups)} \quad (\text{SS3})$$

$$s_G = 0.18$$

$$\beta = 0.99 \quad (\text{SS4})$$

$$\bar{g} = 1.005 \quad (\text{SS5})$$

$$\bar{q} = 1 \quad (\text{SS6})$$

$$\Delta = 0.025$$

$$\bar{\omega} = 0.479 \text{ (preliminary, from GIMF)} \quad (\text{SS7})$$

$$\bar{\sigma} = 0.452 \text{ (preliminary, from GIMF)} \quad (\text{SS8})$$

$$\bar{\xi} = 0.357 \text{ (preliminary, from GIMF)} \quad (\text{SS9})$$

12.1.2 Derived

$$\bar{r} = \frac{\bar{g}}{\beta} \quad (\text{SS10})$$

$$\alpha = 1 - \bar{\mu}_p s_L \quad (\text{SS11})$$

$$\bar{m}\bar{c} = \frac{1}{\bar{\mu}_p} \quad (\text{SS12})$$

$$\bar{z} = \frac{\ln(\bar{\omega}) + \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}} \quad (\text{SS13})$$

$$\bar{\Gamma} = \frac{1}{2} \text{erfc} \left(\frac{\bar{\sigma} - \bar{z}}{\sqrt{2}} \right) + \bar{\omega} \left(1 - \frac{1}{2} \text{erfc} \left(\frac{-\bar{z}}{\sqrt{2}} \right) \right) \quad (\text{SS14})$$

$$\bar{G} = \frac{1}{2} \text{erfc} \left(\frac{\bar{\sigma} - \bar{z}}{\sqrt{2}} \right) \quad (\text{SS15})$$

$$f(\bar{\omega}) = \frac{1}{\sqrt{2\pi\bar{\omega}\bar{\sigma}}} \exp \left\{ -\frac{1}{2} \bar{z}^2 \right\} \quad (\text{SS16})$$

$$\bar{\Gamma}^\omega = 1 - \frac{1}{2} \text{erfc} \left(\frac{-\bar{z}}{\sqrt{2}} \right) \quad (\text{SS17})$$

$$\bar{G}^\omega = \bar{\omega} f(\bar{\omega}) \quad (\text{SS18})$$

$$\bar{\lambda} = \frac{\bar{\Gamma}^\omega}{\bar{\Gamma}^\omega - \bar{\xi} \bar{G}^\omega} \quad (\text{SS19})$$

$$\overline{ret}_k = \frac{\bar{r} \bar{\lambda}}{1 - \bar{\Gamma} + \bar{\lambda} (\bar{\Gamma} - \bar{\xi} \bar{G})} \quad (\text{SS20})$$

$$\bar{r}_k = \overline{ret}_k - 1 + \Delta \quad (\text{SS21})$$

$$\bar{\Gamma}^{\omega\omega} = -f(\bar{\omega}) \quad (\text{SS22})$$

$$\bar{\Gamma}^{\omega\sigma} = f(\bar{\omega}) \bar{\omega} (\bar{z} - \bar{\sigma}) \quad (\text{SS23})$$

$$\bar{\Gamma}^\sigma = -\bar{\omega}^2 \bar{\sigma} f(\bar{\omega}) \quad (\text{SS24})$$

$$\bar{G}^{\omega\omega} = -f(\bar{\omega}) \frac{\bar{z}}{\bar{\sigma}} \quad (\text{SS25})$$

$$\bar{G}^{\omega\sigma} = \frac{\bar{\omega}}{\bar{\sigma}} f(\bar{\omega}) [\bar{z} (\bar{z} - \bar{\sigma}) - 1] \quad (\text{SS26})$$

$$\bar{G}^\sigma = -\bar{\omega}^2 f(\bar{\omega}) \bar{z} \quad (\text{SS27})$$

$$A = \alpha^{-\alpha} (1 - \alpha)^{-(1-\alpha)} \quad (\text{SS28})$$

$$\bar{w} = \left(\frac{\bar{m}\bar{c}}{A (\bar{r}_k)^\alpha} \right)^{\frac{1}{1-\alpha}} \quad (\text{SS29})$$

$$\bar{Y} = \frac{\bar{w}\bar{L}}{(1-\alpha)\bar{m}\bar{c}} \quad (\text{SS30})$$

$$\bar{k} = \frac{\alpha\bar{m}\bar{c}\bar{Y}}{\bar{r}_k} \quad (\text{SS31})$$

$$\bar{K} = \bar{k}\bar{g} \quad (\text{SS32})$$

$$\bar{I} = \frac{(\bar{g} + \Delta - 1) \bar{K}}{\bar{g}} \quad (\text{SS33})$$

$$\overline{GOV} = s_g \bar{Y} \quad (\text{SS34})$$

$$\bar{C} = \bar{Y} - \bar{I} - \overline{GOV} \quad (\text{SS35})$$

$$\bar{n} = \bar{K} \left(1 - \frac{\overline{ret}_k}{\bar{r}_k} (\bar{\Gamma} - \bar{\xi}\bar{G}) \right) \quad (\text{SS36})$$

$$d = \frac{\bar{r}}{\bar{g}} - 1 + \frac{\bar{K}}{\bar{g}\bar{n}} (\overline{ret}_k (1 - \bar{\xi}\bar{G}) - \bar{r}) \quad (\text{SS37})$$

12.2 Demand Block

$$\hat{\lambda}_t = \hat{u}_t + E_t \left(\hat{\lambda}_{t+1} - \hat{\pi}_{t+1}^p - \hat{g}_{t+1} \right) \quad (\text{D1})$$

$$\hat{S}_t^c - \hat{H}_t = \hat{\lambda}_t \quad (\text{D2})$$

$$\hat{H}_t = \frac{1}{1 - \frac{\nu}{g}} \hat{C}_t - \frac{\frac{\nu}{g}}{1 - \frac{\nu}{g}} \left(\hat{C}_{t-1} - \hat{g}_t \right) \quad (\text{D3})$$

$$\bar{K} \hat{K}_t = (1 - \Delta) \frac{\bar{K}}{g} \left(\hat{K}_{t-1} - \hat{g}_t \right) + \bar{I} \hat{I}_{t-1} \quad (\text{D4})$$

$$\hat{q}_t = \phi_I \left(\hat{I}_t - \hat{I}_{t-1} \right) - \beta \phi_I \left(E_t \hat{I}_{t+1} - \hat{I}_t \right) + \hat{S}_t^{inv} \quad (\text{D5})$$

$$\widehat{ret}_{k,t} = \frac{1 - \Delta}{1 - \Delta + \bar{r}_k} E_t \hat{q}_{t+1} - \hat{q}_t + \frac{\bar{r}_k}{1 - \Delta + \bar{r}_k} E_t \hat{r}_{k,t+1} \quad (\text{D6})$$

$$\hat{r}_{k,t} = \sigma_a \hat{u}_t \quad (\text{D7})$$

$$\begin{aligned} 0 = & \left(\frac{\bar{K}}{\bar{n}} - 1 \right) \left(\widehat{ret}_{k,t}^{m1} - \hat{r}_t^{m1} \right) - \left(\hat{q}_{t-1} + \hat{K}_{t-1} - \hat{n}_{t-1} \right) \\ & + \left(\frac{\bar{K}}{\bar{n}} - 1 \right) \frac{\bar{\Gamma}^\omega - \bar{\xi} \bar{G}^\omega}{\bar{\Gamma} - \bar{\xi} \bar{G}} \bar{\omega} \hat{\omega}_t + \left(\frac{\bar{K}}{\bar{n}} - 1 \right) \frac{\bar{\Gamma}^\sigma - \bar{\xi} \bar{G}^\sigma}{\bar{\Gamma} - \bar{\xi} \bar{G}} \bar{\sigma} \hat{\sigma}_t \end{aligned} \quad (\text{D8})$$

$$0 = \bar{\lambda} \left(\widehat{ret}_{k,t} - \hat{r}_t \right) - (1 - \bar{\Gamma}) \frac{\bar{r}^k}{\bar{r}} \left(\frac{\bar{\Gamma}^{\omega\omega} - \bar{\lambda} \left(\bar{\Gamma}^{\omega\omega} - \bar{\xi} \bar{G}^{\omega\omega} \right)}{\bar{\Gamma}^\omega} \right) \bar{\omega} E_t \hat{\omega}_{t+1} \quad (\text{D9})$$

$$+ \frac{\bar{r}^k}{\bar{r}} \left[-\Gamma^\sigma + \bar{\lambda} \left(\bar{\Gamma}^\sigma - \bar{\xi} \bar{G}^\sigma \right) - (1 - \bar{\Gamma}) \left(\frac{\bar{\Gamma}^{\omega\sigma} - \bar{\lambda} \left(\bar{\Gamma}^{\omega\sigma} - \bar{\xi} \bar{G}^{\omega\sigma} \right)}{\bar{\Gamma}^\omega} \right) \right] \bar{\sigma} E_t \hat{\sigma}_{t+1}$$

$$\begin{aligned} & \bar{n} (1 + d) (\hat{n}_t + \hat{g}_t) \quad (\text{D10}) \\ = & \frac{\bar{r}}{g} (\bar{n} - \bar{K}) \hat{r}_t^{m1} + \frac{\bar{r}}{g} \bar{n} \hat{n}_{t-1} + \frac{\bar{K}}{g} \left(\widehat{ret}_k (1 - \bar{\xi} \bar{G}) - \bar{r} \right) \left(\hat{q}_{t-1} + \hat{K}_{t-1} \right) \\ & + \frac{\bar{K}}{g} \widehat{ret}_k (1 - \bar{\xi} \bar{G}) \left(\widehat{ret}_{k,t}^{m1} \right) + \frac{\bar{K}}{g} \widehat{ret}_k \bar{\xi} \left(\bar{G}^\omega \bar{\omega} \hat{\omega}_t + \bar{G}^\sigma \bar{\sigma} \hat{\sigma}_t \right) \end{aligned}$$

$$\widehat{ret}_{k,t}^{m1} = \widehat{ret}_{k,t-1} \quad (\text{D11})$$

$$\hat{r}_t^{m1} = \hat{r}_{t-1} \quad (\text{D12})$$

$$\hat{\psi}_t^w = \delta_w \hat{\psi}_{t-1}^w + (1 - \delta_w) \hat{v}_{t-1}^w - \hat{\varepsilon}_t^{\pi^*} \quad (\text{D13})$$

$$E_t \hat{v}_{t+1}^w = \hat{v}_t^w + \frac{(1 - \delta_w \beta)^2}{(\delta_w \beta)^2} \frac{\delta_w}{1 - \delta_w} \hat{\psi}_t^w - \frac{(1 - \delta_w \beta)^2}{(\delta_w \beta)^2} \frac{\delta_w}{1 - \delta_w} \hat{\pi}_t^w + \frac{(1 - \delta_w \beta)^2}{(\delta_w \beta)^2} \frac{\widehat{mrs}_t - \hat{w}_t + \hat{\mu}_t^w}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)} \quad (\text{D14})$$

$$E_t \hat{\pi}_{t+1}^w = \hat{\pi}_t^w \left(\frac{2}{\beta} - \delta_w \right) + \hat{v}_t^w ((1 - \delta_w)(1 + \delta_w)) \quad (\text{D15})$$

$$+ \hat{\psi}_t^w \left(\delta_w (1 + \delta_w) - \frac{2}{\beta} \right) - \frac{2(1 - \delta_w)(1 - \delta_w \beta)}{(\delta_w \beta)} \frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}_t^w)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)}$$

$$\hat{w}_t = \hat{w}_{t-1} - \hat{g}_t + \hat{\pi}_t^w - \hat{\pi}_t^p \quad (\text{D16})$$

$$\widehat{mrs}_t = \frac{1}{\eta} \hat{L}_t - \frac{\bar{\sigma}^w}{\eta} \frac{\delta_w}{1 - \delta_w} \hat{\pi}_t^w + \frac{\bar{\sigma}^w}{\eta} \frac{\delta_w}{1 - \delta_w} \hat{\psi}_t^w + \hat{S}_t^L - \hat{\lambda}_t \quad (\text{D17})$$

12.3 Supply Block

$$\hat{\psi}_t^p = \delta_p \hat{\psi}_{t-1}^p + (1 - \delta_p) \hat{v}_{t-1}^p - \hat{\varepsilon}_t^{\pi^*} \quad (\text{S1})$$

$$E_t \hat{v}_{t+1}^p = \hat{v}_t^p + \frac{(1 - \delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\delta_p}{1 - \delta_p} \hat{\psi}_t^p - \frac{(1 - \delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\delta_p}{1 - \delta_p} \hat{\pi}_t^p + \frac{(1 - \delta_p \beta)^2}{(\delta_p \beta)^2} \frac{\widehat{mc}_t + \hat{\mu}_t^p}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)} \quad (\text{S2})$$

$$E_t \hat{\pi}_{t+1}^p = \hat{\pi}_t^p \left(\frac{2}{\beta} - \delta_p \right) + \hat{v}_t^p ((1 - \delta_p)(1 + \delta_p)) \quad (\text{S3})$$

$$+ \hat{\psi}_t^p \left(\delta_p (1 + \delta_p) - \frac{2}{\beta} \right) - \frac{2(1 - \delta_p)(1 - \delta_p \beta)}{(\delta_p \beta)} \frac{(\widehat{mc}_t + \hat{\mu}_t^p)}{(1 + \phi_p \bar{\mu}_p \bar{\sigma}_p)}$$

$$\widehat{mc}_t = (1 - \alpha) \hat{w}_t + \alpha \hat{r}_t^k \quad (\text{S4})$$

$$\hat{L}_t = \widehat{mc}_t - \hat{w}_t + \hat{Y}_t \quad (\text{S5})$$

$$\hat{k}_t = \widehat{mc}_t - \hat{r}_t^k + \hat{Y}_t \quad (\text{S6})$$

12.4 Aggregate Relationships

$$\hat{i}_t = \theta^{int} (\hat{i}_{t-1} - \hat{\varepsilon}_t^{\pi^*}) + (1 - \theta^{int} + \theta^\pi) E_t \left(\frac{\hat{\pi}_{t+3}^p + \hat{\pi}_{t+2}^p + \hat{\pi}_{t+1}^p + \hat{\pi}_t^p}{4} \right) \quad (\text{A1})$$

$$+ \theta^y E_t \left(\frac{\hat{Y}_{t+3} - \hat{Y}_{t-1}}{4} \right) + \frac{\hat{S}_t^{int}}{4}$$

$$\hat{r}_t = \hat{i}_t - E_t \hat{\pi}_{t+1}^p \quad (\text{A2})$$

$$\bar{Y} \hat{Y}_t = \bar{C} \hat{C}_t + \bar{I} \hat{I}_t + \overline{GOV} \hat{S}_t^{gov} \quad (\text{A3})$$

$$\hat{k}_t = \hat{K}_{t-1} + \hat{u}_t - \hat{g}_t \quad (\text{A4})$$

12.5 Shocks

12.5.1 Unit Roots

$$\hat{g}_t = \hat{g}_t^{gr} + \hat{g}_t^{iid} \quad (\text{U1})$$

$$\hat{g}_t^{gr} = \rho_g \hat{g}_{t-1}^{gr} + \hat{\varepsilon}_t^{gr} \quad (\text{U2})$$

$$\hat{g}_t^{iid} = \hat{\varepsilon}_t^{iid} \quad (\text{U3})$$

$$\hat{\pi}_t^* = \hat{\pi}_{t-1}^* + \hat{\varepsilon}_t^{\pi^*} \quad (\text{U4})$$

12.5.2 Stationary

$$\hat{S}_t^c = \rho^c \hat{S}_{t-1}^c + \hat{\varepsilon}_t^c \quad (\text{X1})$$

$$\hat{S}_t^L = \rho^L \hat{S}_{t-1}^L + \hat{\varepsilon}_t^L \quad (\text{X2})$$

$$\hat{S}_t^{inv} = \rho^{inv} \hat{S}_{t-1}^{inv} + \hat{\varepsilon}_t^{inv} \quad (\text{X3})$$

$$\hat{S}_t^{gov} = \rho^{gov} \hat{S}_{t-1}^{gov} + \hat{\varepsilon}_t^{gov} \quad (\text{X4})$$

$$\hat{S}_t^{int} = \rho^{int} \hat{S}_{t-1}^{int} + \hat{\varepsilon}_t^{int} \quad (\text{X5})$$

$$\hat{\sigma}_t = \rho^\sigma \hat{\sigma}_{t-1} + \hat{\varepsilon}_t^\sigma \quad (\text{X6})$$

$$\hat{\xi}_t = \rho^\xi \hat{\xi}_{t-1} + \hat{\varepsilon}_t^\xi \quad (\text{X7})$$

$$\hat{\mu}_t^p = \hat{\varepsilon}_t^{\mu^p} \quad (\text{X8})$$

$$\hat{\mu}_t^w = \hat{\varepsilon}_t^{\mu^w} \quad (\text{X9})$$

13 REPORTING VARIABLES

- Remember that the $\hat{\cdot}$ -variables are the ones that come from the computer code.

13.1 Real Variables - Growth Rates

- Real GDP Growth:

$$DOT_GDP_t = \ln Y_t - \ln Y_{t-1} = \ln \check{Y}_t - \ln \check{Y}_{t-1} + \ln g_t = \hat{Y}_t - \hat{Y}_{t-1} + \hat{g}_t + \ln(\bar{g})$$

- Real Consumption Growth:

$$DOT_C_t = \ln C_t - \ln C_{t-1} = \ln \check{C}_t - \ln \check{C}_{t-1} + \ln g_t = \hat{C}_t - \hat{C}_{t-1} + \hat{g}_t + \ln(\bar{g})$$

- Real Investment Growth:

$$DOT_INV_t = \ln I_t - \ln I_{t-1} = \ln \check{I}_t - \ln \check{I}_{t-1} + \ln g_t = \hat{I}_t - \hat{I}_{t-1} + \hat{g}_t + \ln(\bar{g})$$

- Real Wage Growth:

$$DOT_W_t = \ln w_t - \ln w_{t-1} = \ln \check{w}_t - \ln \check{w}_{t-1} + \ln g_t = \hat{w}_t - \hat{w}_{t-1} + \hat{g}_t + \ln(\bar{g})$$

13.2 Nominal Variables - Growth Rates

- Inflation Growth:

$$DOT_PAI_t = \ln(\pi_t^p) - \ln(\pi_{t-1}^p) = \ln(\check{\pi}_t^p) - \ln(\check{\pi}_{t-1}^p) + \ln(\varepsilon_t^{\pi^*}) = \hat{\pi}_t^p - \hat{\pi}_{t-1}^p + \hat{\varepsilon}_t^{\pi^*}$$

- Nominal Interest Rate Growth:

$$DOT_INT_t = \ln(i_t) - \ln(i_{t-1}) = \ln(\check{i}_t) - \ln(\check{i}_{t-1}) + \ln(\varepsilon_t^{\pi^*}) = \hat{i}_t - \hat{i}_{t-1} + \hat{\varepsilon}_t^{\pi^*}$$

13.3 Real Variables - Stationary

- Labor Supply:

$$NODOT_L_t = \ln(L_t) - \ln(\bar{L}) = \hat{L}$$